MA214 Solutions to Test 2

1. $y'' + 3y' + 2y = u_\pi(t)$ & $y(0) = 0 = y'(0)$

$$(s^2+3s+2)(\mathcal{L}y)(s) = \frac{e^{-\pi s}}{s}$$

$$(\mathcal{L}y)(s) = \frac{e^{-\pi s}}{s(s+2)(s+1)} = e^{-\pi s} H(s)$$

$$H(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} = \frac{1}{s(s+2)(s+1)}$$

$$1 = A(s^2+3s+2) + Bs(s+1) + C(s+2)s$$

$$= (A+B+C)s^2 + (3A+B+2C)s + 2A$$

$$\begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2} \\ C = -1 \end{cases} \qquad \begin{array}{r} B + C = -\frac{1}{2} \\ B + 2C = -3/2 \\ \hline C = -1 \end{array}$$

$$(\mathcal{L}^{-1}H)(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$

$$y(t) = u_\pi(t)\,\frac{1}{2}\left(1 + e^{-2(t-\pi)} - 2e^{-(t-\pi)}\right)$$

4. $y'' + 2y' + 2y = t$

Homog. ODE: $r^2 + 2r + 2 = 0$

roots: $\frac{-2 \pm [4-8]^{\frac{1}{2}}}{2} = -1 \pm i$

2 solns: $y_1 = e^{-t}\cos t$ $\quad y_2 = e^{-t}\sin t$

Particular Soln:

$$\left.\begin{array}{l} y_p(t) = At + B \\ y_p'(t) = A \\ y_p''(t) = 0 \end{array}\right\} \quad y_p(t) = \frac{1}{2}(t-1)$$

Substitute: $2A + 2At + 2B = t$

$2(A+B) = 0 \quad A = \frac{1}{2} \quad B = -\frac{1}{2}$

T2-2

General Real Soln:

$$y(t) = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t + \tfrac{1}{2}(t-1)$$

3. $y'' + 4y' + 4y = t^{-2}e^{-2t}$

Homog. ODE $\quad r^2+4r+4 = (r+2)^2$ repeated root.

$$y_1(t) = e^{-2t} \qquad y_2(t) = te^{-2t}$$

Particular Soln: $W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t}(1-2t) \end{vmatrix}$

$$= e^{-4t}(1-2t) + 2te^{-4t}$$

$$= e^{-4t}$$

$$u_1' = -\frac{gy_2}{W} = -\frac{e^{-2t}}{t^2}\cdot\frac{te^{-2t}}{e^{-4t}} = -\frac{1}{t} \qquad u_1(t) = -\ln t$$

$$u_2' = \frac{gy_1}{W} = \frac{e^{-2t}}{t^2}\cdot\frac{e^{-2t}}{e^{-4t}} = \frac{1}{t^2} \qquad u_2(t) = -\frac{1}{t}$$

$$y_p(t) = -\ln t\cdot e^{-2t} - \frac{1}{t}\,te^{-2t} = -\ln t\cdot e^{-2t}\underbrace{-e^{-2t}}_{\text{soln. to homog ODE}}$$

General Soln: $y(t) = C_1e^{-2t} + C_2te^{-2t} - (\ln t)e^{-2t}$.

2. $y'' + \omega_0^2 y = \frac{F_0}{m}t \quad$ Soln. of homog ODE: $C_1\cos\omega_0 t + C_2\sin\omega_0 t$

Guess $y_p(t) = At + B \quad y_p'(t) = A \quad y_p''(t) = 0$

So: $y_p'' + \omega_0^2 y_p = \omega_0^2 At + B\omega_0^2 = \frac{F_0}{m}t \Rightarrow B = 0 \quad A = \frac{F_0}{\omega_0^2 m}$

General soln: $y(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{\omega_0^2 m}t$

Initial Conditions:

$$y'(t) = -\omega_0 C_1\sin\omega_0 t + C_2\omega_0\cos\omega_0 t + \frac{F_0}{\omega_0^2 m}$$

$y(0) = C_1 = 0 \quad y'(0) = C_2\omega_0 + \frac{F_0}{\omega_0^2 m} = 0$.

$$y(t) = \frac{F_0}{\omega_0^2 m}\left(t - \frac{1}{\omega_0}\sin\omega_0 t\right)$$