MA/PHY 506 Fall 2012
Final Exam - 100 Points
14 December 2012

INSTRUCTIONS: PLEASE WORK ALL FIVE PROBLEMS BELOW. NO BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	20	
2	25	
3	10	
4	25	
5	20	
TOTAL	100	

Problem 1. (20 points.) Suppose that a linear transformation T acts on the standard basis vectors $\{e_1, e_2\}$ for R^2 by

$$Te_1 = 2e_1 + 2e_2, \tag{1}$$

$$Te_2 = e_1 + 3e_2, \tag{2}$$

i. Write the matrix representation of T relative to this basis.

ii. Compute T^{-1}, if it exists.

iii. Find the characteristic equation for T and the eigenvalues.

iv. Find the corresponding normalized eigenvectors $\{f_1, f_2\}$.

iv. Construct the matrix S with entries $S_{ij} = \langle e_i, f_j \rangle$. Show that $T' = S^{-1}TS$ is diagonal.

i. Matrix rep: $T_{ij} = \langle \hat{e}_i, T\hat{e}_j \rangle \quad T_{11} = 2 \quad T_{21} = 2 \quad T_{12} = 1 \quad T_{22} = 3$

$$T = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$$

ii. $\det T = 2 \cdot 3 - 2 \cdot 1 = 6 - 2 = 4 \neq 0$ so T is nonsingular & T^{-1} exists.

$$T^{-1} = \frac{1}{\det T}\begin{pmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$$

Check: $TT^{-1} = \frac{1}{4}\begin{pmatrix} 6-2 & -2+2 \\ 6-6 & -2+6 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = I_2$.

iii. $\det(T - \lambda I_2) = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2 = 6 - 5\lambda + \lambda^2 - 2$

$$= \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$$

$\sigma(T) = \{1, 4\}$ ev. Each has multiplicity 1. So there is a basis of eigenvectors since we are on $\mathbb{R}^2$. T is diagonalizable even though it isn't symmetric.

iv. Eigenvectors $\underline{\lambda = 1}$ $\quad (T - I)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ 2(a+b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = -b$

$\hat{f}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ check $T\hat{f}_1 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \hat{f}_1$.

$\underline{\lambda = 4}$ $\quad (T - 4)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2a+b \\ 2a-b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $b = 2a$

so $\hat{f}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{\text{check}}$ $T\hat{f}_2 = \begin{pmatrix} 4 \\ 8 \end{pmatrix}\frac{1}{\sqrt{5}} = 4\hat{f}_2$

v. $S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{pmatrix}$. Since $\langle \hat{f}_1, \hat{f}_2 \rangle \neq 0$ S isn't orthogonal

but $\det S = \frac{3}{\sqrt{10}}$ so S^{-1} exists. $S^{-1} = \frac{\sqrt{10}}{3}\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2} \end{pmatrix}$

$$= \begin{pmatrix} \frac{2\sqrt{2}}{3} & -\sqrt{2}/3 \\ \sqrt{5}/3 & \sqrt{5}/3 \end{pmatrix}$$

Calculate:

$$S^{-1} T S = S^{-1} \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{pmatrix} = S^{-1} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{4}{\sqrt{5}} \\ -\frac{1}{\sqrt{2}} & 8/\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ \sqrt{5}/3 & \sqrt{5}/3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 4/\sqrt{5} \\ -1/\sqrt{2} & 8/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} \frac{2\sqrt{2}}{3} & -\sqrt{2}/3 \\ \sqrt{5}/3 & \sqrt{5}/3 \end{pmatrix} = S^{-1}$$

Problem 2. (25 points.) The Hermite equation is

$$y'' - xy' + qy = 0,$$

where $q \geq 0$.

i. Give the definition of the Wronskian and compute it for any two independent solutions of the Hermite ODE. This is Abel's formula

ii. Find two linearly independent solutions, and their Wronskian, when $q = 0$.

iii. What happens when q is a positive integer? Write out the first three Hermite polynomials. You need not normalize them.

i. Wronskian: 2 fncs u_1 & u_2

$$W(u_1,u_2)(x) = u_1(x)u_2'(x) - u_1'(x)u_2(x)$$

Abel's formula:

$$W(y_1,y_2)(x) = C_0 e^{-\int^x p(s)ds} = C_0 e^{x^2/2}$$

$$p(s) = -s.$$

If y_1 & y_2 are LI $\Rightarrow C_0 \neq 0$.

ii. $q=0$: $y'' - xy' = 0$ one soln. $y_1(x) = 1$

second soln: $w = y_2'$ $\quad w' - xw = 0 \quad \frac{dw}{w} = x$

so $\log w = x^2/2 + C \quad w(x) = y_2'(x) = Ce^{x^2/2}$

$$y_2(x) = \int_0^x e^{+s^2/2}\,ds. \quad (\text{set } C=1)$$

Check: $y_2'(x) = e^{x^2/2} \quad y_2''(x) = xe^{x^2/2}$

so $y_2'' - y_2'x = 0$.

$$W(y_1,y_2)(x) = \begin{vmatrix} 1 & \int^x e^{s^2/2}ds \\ 0 & e^{x^2/2} \end{vmatrix} = e^{x^2/2}$$

just as above with $C_0 = 1$.

iii. $x_0 = 0$ ordinary point: $y(x) = \sum_{j=0}^{\infty} a_j x^j$

$$y'(x) = \sum_{j=0}^{\infty} j a_j x^{j-1}$$

$$y''(x) = \sum_{j=0}^{\infty} j(j-1)a_j x^{j-2}$$

ODE $= \sum_{j=0}^{\infty} \left[j(j-1) a_j x^{j-2} + (q-j) a_j x^j \right] = 0$

Re-index 1^{st} term:

$m = j-2$

$j = m+2$

$$\sum_{m=-2}^{\infty} (m+2)(m+1) a_{m+2} x^m = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

Substitute:

$$\sum_{j=0}^{\infty} \left[(j+2)(j+1) a_{j+2} + (q-j) a_j \right] x^j = 0$$

Recursion relation:

$$\boxed{a_{j+2} = \frac{(j-q)\, a_j}{(j+1)(j+2)}}$$

If $a_0 \neq 0$ & $a_1 = 0$ then j is even & this gives a poly. soln. if q is even

If $a_0 = 0$ & $a_1 \neq 0$ then j is odd & this gives a poly. soln if q is odd.

Hermite poly:

$q=0$: $a_0 \neq 0$ $a_1 = 0$ $a_2 = 0$, $a_4 = 0$, ...

- $H_0(x) = 1$ even

$q=1$: j odd $a_0 = 0$ $a_1 \neq 0$ $a_3 = 0$ ...

- $H_1(x) = x$ odd

$q=2$: $a_0 \neq 0$ $a_1 = 0$ $a_2 \neq 0$, $a_4 = 0$ ...

$a_0 \neq 0$, $a_2 = -\frac{2}{2} a_0$ if $a_0 = 1$ $a_2 = -1$

- $H_2(x) = 1 - x^2$

Problem 3. (10 points.) Solve the ODE: $xy' + 2y = x$, with the condition $y(1) = 1$.

$$y' + \frac{2}{x} y = 1$$

Integrating factor: $\mu(x) = e^{\int^x p(s)\,ds} = e^{\int^x 2\frac{ds}{s}} = e^{2\log x} = x^2.$

$p(s) = \frac{2}{s}$

$(\mu y)'(x) = \mu q = x^2 \Rightarrow \mu(x) y(x) = \frac{x^3}{3} + C$

General soln. $\boxed{y(x) = \frac{x}{3} + \frac{C}{x^2}}$

Check: $y'(x) = \frac{1}{3} - \frac{2C}{x^3}$

$\frac{2}{x} y(x) = \frac{2}{3} + \frac{2C}{x^3}$

Add: $y' + \frac{2}{x} y = 1$. ✓

Initial Cond:

$y(1) = \frac{1}{3} + C = 1 \quad C = \frac{2}{3}$

$$\boxed{y(x) = \frac{x}{3} + \frac{2}{3x^2}}$$

Problem 4. (25 points.) Let $Lu = -u''$ be an operator on $L^2([0,\ell])$ with the boundary conditions:

$$u(0) = 0, \quad u'(\ell) = 0.$$

i. Show that L is hermitian on the subspace of functions satisfying the boundary conditions.

ii. Find the eigenvalues and eigenfunctions of L.

iii. Write an eigenfunction expansion for the Green's function for L.

iv. Use the four properties of the Green's function in order to express the Green's function in terms of two solutions to $Lw = 0$.

i. $L^2([0,\ell])$ Compute $\langle w, Lu \rangle$ when w & u satisfy the BC

$$\langle w, Lu \rangle = -\int_0^\ell \bar{w}(x) u''(x) = -\bar{w}(x)u'(x)\Big|_0^\ell + \int_0^\ell \bar{w}'(x)u'(x)$$

$$= -\bar{w}(\ell)\underbrace{u'(\ell)}_{=0} + \underbrace{\bar{w}(0)}_{=0}u'(0) + \int_0^\ell \bar{w}'(x)u'(x)$$

$$= \bar{w}'(x)u(x)\Big|_0^\ell - \int_0^\ell \bar{w}''(x)u(x)$$

$$= \underbrace{\bar{w}'(\ell)}_{=0}u(\ell) - \bar{w}'(0)\underbrace{u(0)}_{=0} + \langle Lw, u\rangle. \checkmark \text{ hermitian}$$

ii. Eigenvalues & eigenfncs. $Lu = \lambda u$, u satisfies BC.

$$-u'' = \lambda u, \quad u(x) = C_\lambda^{(1)} e^{i\sqrt{\lambda}x} + C_\lambda^{(2)} e^{-i\sqrt{\lambda}x}$$

BC: $u(0) = C_\lambda^{(1)} + C_\lambda^{(2)} = 0 \qquad C_\lambda^{(1)} = -C_\lambda^{(2)}$

$$u'(x) = i\sqrt{\lambda}\left(e^{i\sqrt{\lambda}x} + e^{-i\sqrt{\lambda}x}\right)C_\lambda^{(1)}$$

$$u'(\ell) = i\sqrt{\lambda}\left(2\cos\sqrt{\lambda}\ell\right)C_\lambda^{(1)} = 0$$

$$\cos\sqrt{\lambda}\ell = 0 \iff \sqrt{\lambda}\ell = \frac{2k+1}{2}\pi \quad \text{odd multiples of } \pi$$

$$\boxed{\begin{aligned}\lambda_k &= \left[\frac{2k+1}{2\ell}\pi\right]^2 \\ u_k(x) &= \sqrt{\frac{2}{\ell}}\sin(\sqrt{\lambda_k}x)\end{aligned}}$$

Check: $\displaystyle\int_0^\ell \sin^2\sqrt{\lambda_k}x\,dx = \frac{1}{2}\int_0^\ell (1 - \cos(2\sqrt{\lambda_k}x))dx = \ell/2.$

iii.) $G_L(x,y) = \sum_{k=0}^{\infty} \frac{2\sin(\sqrt{\lambda_k}x)\sin(\sqrt{\lambda_k}y)}{l\lambda_k}$, $\lambda_k = \left[\frac{2k+1}{2l}\pi\right]^2$

iv.) Solve $Lw=0$, $w(x) = C_1 + C_2 x$ basis : $\{1, x\}$

$w'(x) = C_2$

$w''(x) = 0$.

$u_1(x)$ satisfies $u_1(0)=0$. $u_1(x) = C_1 + C_2 x$

$u_1(0) = C_1 = 0 \Rightarrow u_1(x) = x$

$u_2(x)$ satisfies $u_2'(l)=0$

$u_2'(x) = C_2 = 0$ so $u_2(x) = 1$

$$G_L(x,y) = \begin{cases} \frac{u_1(x)u_2(y)}{p(y)W(y)} & 0 \le x < y \le l \\ \frac{u_1(y)u_2(x)}{p(y)W(y)} & 0 \le y < x \le l \end{cases}$$

$W(u_1,u_2)(y) = \begin{vmatrix} y & 1 \\ 1 & 0 \end{vmatrix} = -1$ $p(y) = -1$

$$G_L(x,y) = \begin{cases} x & 0 \le x < y \le l \\ y & 0 \le y < x \le l \end{cases}$$

$\begin{cases} G_L(0,y) = 0 \\ G'(l,y) = \frac{d}{dx} y = 0 \end{cases}$ since $y<l$ so use the 2^{nd} form.

so G satisfies BC

$\begin{cases} G_L(x,0) = 0 \\ \left.\frac{d}{dy} G(x,y)\right|_{y=l} = G'(x,l) = 0 \end{cases}$

$G''(x,y) = 0$ $x \neq y$ in either variable

jump: $G'(y+\varepsilon, y) - G'(y-\varepsilon, y) = -1$ ✓

Problem 5. (20 points.) Find the most general solution to the inhomogeneous problem

$$y'' + 3y' - 4y = xe^x.$$

Use this to solve the initial-value problem: $y(0) = 1$ and $y'(0) = 0$.

i. $r^2 + 3r - 4 = 0 = (r+4)(r-1) = 0$

$y_1(x) = e^{-4x}$ $\quad y_2(x) = e^x$ $\quad$ basis of soln. space.

$W(y_1, y_2)(x) = \begin{vmatrix} e^{-4x} & e^x \\ -4e^{-4x} & e^x \end{vmatrix} = 5e^{-3x} \neq 0.$

ii. Variation of Parameters for $y_p(x)$.

$c_1'(x) = \frac{-y_2(x)xe^x}{W(x)} \Rightarrow - \int \frac{xe^{2x}}{5e^{-3x}} dx = -\frac{1}{5} \int xe^{5x} dx$

$\int xe^{5x} dx = \frac{1}{5} x e^{5x} - \int \frac{1}{5} e^{5x} dx$

$= \frac{1}{5} xe^{5x} - \frac{1}{25} e^{5x} + C$

$c_1(x) = -\frac{1}{25} xe^{5x} + \frac{1}{125} e^{5x} + C$

$c_2'(x) = \frac{y_1(x)xe^x}{W(x)} \Rightarrow c_2(x) = \int \frac{e^{-3x} x}{5e^{-3x}} dx = \frac{1}{5} \int x \, dx$

$c_2(x) = \frac{1}{10} x^2 + D$

$y_p(x) = \left(-\frac{1}{25} xe^{5x} + \frac{1}{125} e^{5x} + C\right) e^{-4x} + \left(\frac{1}{10} x^2 + D\right) e^x$

$= -\frac{1}{25} xe^x + \frac{1}{125} e^x + Ce^{-4x} + \frac{1}{10} x^2 e^x + De^x$

all in soln. subsp.

$\boxed{y_p(x) = -\frac{1}{25} xe^x + \frac{1}{10} x^2 e^x}$

$y_p'(x) = -\frac{1}{25} e^x - \frac{1}{25} xe^x + \frac{1}{5} xe^x + \frac{1}{10} x^2 e^x = -\frac{1}{25} e^x + \frac{4}{25} x e^x + \frac{1}{10} x^2 e^x$

$$y_p''(x) = \left(\frac{3}{25} + \frac{9}{25}x + \frac{1}{10}x^2\right)e^x$$

Substitute: $$y_p'' + 3y_p' - 4y_p = e^x\overbrace{\left[\frac{3}{25} - \frac{3}{25}\right]}^{=0} + xe^x\overbrace{\left[\frac{8}{25} + \frac{12}{25} + \frac{4}{25}\right]}^{=1}$$
$$+ x^2e^x\underbrace{\left[\frac{1}{10} + \frac{3}{10} - \frac{4}{10}\right]}_{=0} = xe^x \quad ✓$$

General soln:

$$\boxed{y(x) = C_1e^{-4x} + C_2e^x + \left(\frac{1}{10}x^2 - \frac{1}{25}x\right)e^x}$$

Initial Conditions: $y(0) = 1$
$y'(0) = 0$

$$y(0) = C_1 + C_2 = 1$$
$$y'(0) = -4C_1 + C_2 - \frac{1}{25} = 0$$
$$5C_2 = \frac{101}{25} \qquad C_2 = \frac{101}{125}$$
$$C_1 = 1 - C_2 = \frac{24}{125}$$

Check

$$y(x) = \frac{24}{125}e^{-4x} + \frac{101}{125}e^x + \left(\frac{1}{10}x^2 - \frac{1}{25}x\right)e^x$$

$y(0) = 1$ ✓

$$y'(x) = \frac{-96}{125}e^{-4x} + \frac{101}{125}e^x + \left(\frac{1}{10}x^2 - \frac{1}{25}x\right)e^x$$
$$+ \left(\frac{1}{5}x - \frac{1}{25}\right)e^x$$

$$y'(0) = \frac{5}{125} - \frac{1}{25} = 0 \ ✓$$