MA/PHY 506.

PS1-1

7.2.2. $(s^2+1)f'(s) + s f(s) = 0$

Separate variables (s, f)

$$(s^2+1) f'(s) = - s f(s) \Rightarrow \frac{df}{f} = - \frac{s}{s^2+1} ds$$

$$\log|f| + C_1 = -\frac{1}{2}\log(s^2+1) + C_2$$

$$\log|f(s)| = \log\left[(1+s^2)^{-1/2}\right] + C$$

$$f(s) = C(s^2+1)^{-1/2}$$

Check:

$$f'(s) = -\tfrac{1}{2} C (s^2+1)^{-3/2}\, 2s$$

$$(s^2+1) f'(s) = - s C (s^2+1)^{-1/2} = - s f(s) \checkmark$$

7.2.3. $\frac{dN}{dt} = -kN^2$ Separate variables (t, N)

$$\int \frac{dN}{N^2} = -\int k\,dt = -kt + C_2$$

$$\parallel \quad -N^{-1} + C_1 = -kt + C_2$$

$$-N^{-1}(t) = -kt + C$$

$$N(t) = \frac{1}{-C+kt}, \quad \text{Check: } N'(t) = \frac{-k}{(kt-C)^2} = -kN^2$$

Initial Cond.: $N(t=0) = N_0 = -\frac{1}{C}$ $\quad C = -1/N_0$

$$N(t) = \frac{N_0}{1+N_0 kt} = N_0\left(1 + t/\tau_0\right)^{-1}$$

so $t \to -\tau_0$, $N(t) \to +\infty$.

7.2.5. $m v' = -k v^n$ Separate (t, v): $\frac{dv}{v^n} = -\frac{k}{m} dt$

$$\frac{v^{-n+1}}{1-n} + C_1 = -\frac{k}{m} t + C_2 \quad \text{provided } n \neq 1.$$

PS1 · 2

If $n=1$, $\int \frac{dv}{v} = \log v + C_1 = -\frac{k}{m} t + C_2$

$$\log v(t) = -\frac{k}{m} t + C$$

$$\boxed{n=1 \quad v(t) = Ce^{-tk/m} = V_0 e^{-kt/m}}$$

If $n \neq 1$, $\frac{v^{1-n}}{1-n} = -\frac{k}{m} t + C$ or $\frac{1}{v^{n-1}} = (n-1)\frac{k}{m} t + C$

$$v(t) = \left[C + (n-1)\frac{k}{m} t\right]^{\frac{-1}{n-1}}$$

$$v(0) = C^{\frac{-1}{n-1}} = V_0 \qquad C = V_0^{-n+1}$$

$$\boxed{v(t) = \left[V_0^{-n+1} + (n-1)\frac{k}{m} t\right]^{\frac{-1}{n-1}}}$$

$$\begin{cases} x(t) = \\ t = f(x) \end{cases}$$

To find v as a fnc. of x distance write

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx} v$$

so the ODE is:

$$m v \frac{dv}{dx} = -k v^n$$

$$\boxed{\frac{dv}{v^{n-1}} = -k m^{-1} dx}$$

if $n \neq 2$, $\frac{v^{-n+2}}{2-n} = -\int \frac{k}{m} dx \qquad \frac{1}{v^{n-2}} = (n-2)\frac{k}{m} x + C$

$$v(x) = \left[C + (n-2)\frac{k}{m} x\right]^{\frac{-1}{n-2}}$$

PS1-3

If $n=1$: $dv = -\frac{k}{m}dx \Rightarrow v(x) = -\frac{k}{m}x + v_0$

$v(x=0) = v_0$ so

$$V(x) = \left[v_0^{2-n} + (n-2)\frac{k}{m}x \right]^{-\frac{1}{n-2}}$$

if $n=2$ $\frac{dv}{v} = -\frac{k}{m}dx$

$\log V = -k/m\,x + c \Rightarrow V(x) = V_0 e^{-k/m\,x}$.

7.2.12 $mv' = mg - bv$ $\quad v' = g - \frac{b}{m}v = g\left(1 - \frac{b}{mg}v\right)$

$$\frac{dv}{g(1-v/v_T)} = dt \qquad \boxed{V_T \equiv \frac{b}{mg}}$$

$$\int \frac{dv}{V_T - v} = g v_T\, dt$$

$$-\log(V_T - v) = g v_T t + c$$

$$V_T - v(t) = Ce^{-b/m\,t}$$

$$V(t) = V_T - Ce^{-b/m\,t}$$

Initial Cond: $V(t=0) = 0 = V_T - C$ so $C = V_T$

$$V(t) = V_T\left(1 - e^{-b/m\,t}\right)$$

Note that $v(t) \to V_T$ as $t \to +\infty$.

7.3.1 $y''' + 2y'' - y' + 2y = 0$.

$r^3 - 2r^2 - r + 2 = 0$. $\quad r=1$ root

$(r-1)(r^2 - r - 2) = 0$ $\quad r=-1$ root

$(r-1)(r+1)(r-2) = 0 \quad r = 1, -1, 2.$

3 solns $\{e^t, e^{-t}, e^{2t}\}$.

7.3.4 $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$ roots: $\dfrac{-2 \pm [4-8]^{\frac{1}{2}}}{2}$

$= -1 \pm \frac{1}{2} 2i = -1 \pm i$

$\{e^{-t} \cos t, e^{-t} \sin t\}$