PHY/MA 506 F 2012 ps4pt2-1

Solutions to PS 4 - part 2.

8.22 Hermite ODE: $Ly = y'' - 2xy' + 2\alpha y$

$p_0(x) = 1 > 0 \quad p_1(x) = -2x$

$w(x) = \frac{1}{p_0(x)} e^{\int \frac{p_1(x)}{p_0(x)} dx} = e^{-x^2}$

look at $w(x) L = w\frac{d^2}{dx^2} - 2xw\frac{d}{dx} + 2\alpha w \quad (*)$

We want $wL = \frac{d}{dx} w \frac{d}{dx} + 2\alpha w$

Check: $\frac{d}{dx} w \frac{d}{dx} = w\frac{d^2}{dx^2} + w'\frac{d}{dx} = w\frac{d^2}{dx^2} - 2xw\frac{d}{dx}$

and this is exactly the 1st two terms of $*$.

Conclusion: $\mathcal{L} = wL$ is hermitian on $L^2(\mathbb{R})$ with the usual inner product acting on functions so $\lim_{M\to\infty} \bar{f}'(x) g(x) e^{-x^2} \Big|_{-M}^{M} = 0$

(necessary so in $\langle f, \mathcal{L}g\rangle = \langle \mathcal{L}f, g\rangle$ the boundary terms vanish)

8.2.5 Suppose $Lu_j = \lambda_j u_j$ with $Lu = u'' - 2xu'$

We saw $\mathcal{L} = wL$ is hermitian so the eigenvalue eqn. is

$wLu_j = \mathcal{L}u_j = \lambda_j w u_j$

$j = 1, 2. \quad \lambda_1 \neq \lambda_2$

$\mathcal{L}$ is hermitian as described above.

Here is an application to the Hermite operator

PS4, p. 2-2

Compute: $\langle (\mathcal{L}-\lambda_1)u_1, u_2\rangle = 0$

$= \langle u_1, (\mathcal{L}-\lambda_1)u_2\rangle = (\lambda_2-\lambda_1)\langle u_1, u_2\rangle$

Note that you need $\mathcal{L}$ here.

Since $\lambda_1 \neq \lambda_2$, we get $\langle u_1, u_2\rangle = 0$.

In general

If $Lu_j = \lambda_j u_j$ on an inner product space with inner product $\langle \cdot, \cdot \rangle$ and $L$ is hermitian with respect to this inner product then

$0 = \langle (L-\lambda_1)u_1, u_2\rangle = \langle u_1, (L-\lambda_1)u_2\rangle = (\lambda_2-\lambda_1)\langle u_1, u_2\rangle$

Provided $\lambda_1$ is real. But ev of hermitian operators are real. Now if $\lambda_1 \neq \lambda_2$ we get

$\langle u_1, u_2\rangle = 0$.

This implies LI since if not there would be a $c_1 \neq 0$ so

$u_2 = c_1 u_1$

then $\langle u_2, u_1\rangle = \bar{c}_1\langle u_1, u_1\rangle = 0$

as $\|u_1\|^2 \neq 0$

we get $\bar{c}_1 = 0$, a contradiction.

Problem 3 $L = -\frac{\partial^2}{\partial x^2}$ on $L^2([0,1])$ with DBC.

Let $f, g$ satisfy DBC & be twice differentiable.

PS 4 pt 2 - 3

$$\text{then } \langle f, Lg\rangle = -\int_0^1 \bar{f} g'' = -\Big( \underbrace{\bar{f} g' \Big|_0^1}_{=0 \text{ since } \bar{f} \text{ satisfies DBC}} - \int_0^1 \bar{f}' g' \Big)$$

$$= \int_0^1 \bar{f}' g' = \underbrace{\bar{f}' g \Big|_0^1}_{=0 \text{ since } g \text{ satisfies DBC}} - \int_0^1 \bar{f}'' g = \langle Lf, g\rangle$$

Now solve $\quad Lu = \lambda u = -u'' \quad (\lambda \geq 0)$

$u'' + \lambda u = 0 \quad$ 2 LI solns $\begin{cases} y_1^\lambda(x) = \cos\sqrt{\lambda}x \\ y_2^\lambda(x) = \sin\sqrt{\lambda}x \end{cases}$

So a general LC is $\quad y(x) = C_1^\lambda y_1^\lambda(x) + C_2^\lambda y_2^\lambda(x)$.

Satisfy the BC:

$y(0) = C_1^\lambda y_1^\lambda(0) + C_2^\lambda y_2^\lambda(0) = C_1^\lambda = 0$

$y(1) = C_2^\lambda \sin\sqrt{\lambda} = 0$

we must have $\sqrt{\lambda} = n\pi$

Eigenvalues: $\quad \lambda_n = (n\pi)^2$

Eigen fncs: $\quad \sin(n\pi x) = u_n(x)$

so

$Lu_n = \lambda_n u_n$ & $u_n$ has DBC