8.22 Hermite ODE: \( L y = y'' - 2xy' + 2xy \)

\[
\begin{align*}
\rho_0(x) &= 1 + x \\
\rho_1(x) &= -2x \\
W(x) &= \frac{1}{\rho_0(x)} \int \frac{\rho_1(x)}{\rho_0(x)} \, dx = e^{-x^2}
\end{align*}
\]

Look at \( W(x) L = \frac{w'' - 2xw' + 2xw}{dx^2} \)

We want \( W L = \frac{d}{dx} \frac{dw}{dx} + 2x \frac{dw}{dx} \)

Check: \( \frac{d}{dx} \frac{dw}{dx} = \frac{w'd - wd''}{dx^2} \frac{dw}{dx} - \frac{2xwd}{dx} \frac{dw}{dx} \frac{dx^2}{dx} \frac{dw}{dx} \frac{dx}{dx} \frac{dx}{dx} \frac{dw}{dx} \)

and this is exactly the two terms of \( \star \).

Conclusion: \( L = W L \) is hermitian on \( L^2(\mathbb{R}) \) with the usual inner product acting on functions so

\[ \lim_{m \to \infty} \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2} \, dx \mid_m^M = 0 \]

(necessary so \( \langle f, Lg \rangle = \langle Lf, g \rangle \) the boundary terms vanish.

8.25 Suppose \( L u_j = a_j u_j \) with \( L u = u'' - 2xu' \).
Here we saw \( L = w L \) is hermitian, so the application eigenvale eqn. is

\( L u_j = a_j w u_j \) to the operator \( \mathcal{L} \) is hermitian as described above.
Compute: \[ \langle (L - \lambda_1) u_1, u_2 \rangle = 0 \]

\[= \langle u_1, (L - \lambda_1) u_2 \rangle = (\lambda_2 - \lambda_1) \langle u_1, u_2 \rangle \]

Note that you need to have:

Since \( \lambda_1 \neq \lambda_2 \), we get \( \langle u_1, u_2 \rangle = 0 \).

In general:

If \( L u = \lambda u \) on an inner product space with inner product \( \langle \cdot, \cdot \rangle \) and \( L \) is hermitian, with respect to this inner product, then

\[ 0 = \langle (L - \lambda_1) u_1, u_2 \rangle = \langle u_1, (L - \lambda_1) u_2 \rangle = (\lambda_2 - \lambda_1) \langle u_1, u_2 \rangle \]

Provided \( \lambda_1 \) is real. But ev of hermitian operators are real. Now if \( \lambda_1 \neq \lambda_2 \) we get \( \langle u_1, u_2 \rangle = 0 \).

This implies \( L \) since if not there would be a \( c 
eq 0 \) so

\[ u_2 = c u_1 \]

Then \( \langle u_1, u_2 \rangle = \bar{c} \langle u_1, u_1 \rangle = 0 \)

as \( \| u_1 \|^2 \neq 0 \)

we get \( c = 0 \), a contradiction.

Problem 3:

\( L = -\frac{d^2}{dx^2} \) on \( L^2([0,1]) \) with OBC.

Let \( f \) satisfy OBC & be twice differentiable.
\[ f(x) = e^{-x^2} \]

\[ \langle f, Lg \rangle = -\int_0^1 f'g' = -\left( f'g \bigg|_0^1 - \int_0^1 f''g \right) \]

\[ = \int_0^1 f'g' = f'g \bigg|_0^1 - \int_0^1 f''g = \langle f, g \rangle \]

Since \( g \) satisfies DBC

Now solve \( Lu = 2u = -u'' \quad (x > 0) \)

\( u'' + 2u = 0 \) \quad 2 Li solutions

\[
\begin{align*}
\text{Li solns} & \quad \frac{y_1}{y_2} = \cos 2\sqrt{x} \\
\text{Li solns} & \quad \frac{y_2}{y_1} = \sin 2\sqrt{x}
\end{align*}
\]

So a general \( L < \) is \( y(x) = c_1 y_1(x) + c_2 y_2(x) \)

Satisify the BC:

\[
\begin{align*}
y(0) &= c_1 y_1(0) + c_2 y_2(0) = 0 \\
y'(0) &= c_1 y_1'(0) + c_2 y_2'(0) = 0
\end{align*}
\]

we must have \( \sqrt{x} = \pi \)

Eigenvalues: \( \lambda_n = (\pi n)^2 \)

Eigen fns: \( \sin (\pi n x) = y_n(x) \)

so

\( Lu_n = \lambda_n u_n \) & \( u_n \) has DBC