PHY / MA 506 (5.1)

Solutions to PS 5

Legendre's ODE eigenvalue problem.

$$Ly = -(1-x^2)y'' + 2xy' = n(n+1)y$$

NOTE: n is not necessarily an integer.

Series: $x_0 = \pm 1$ reg sing pts.

$$y(x) = \sum_{j \geq 0} a_j x^{j+s} \qquad \text{Solve. } \big(L - n(n+1)\big)y = 0$$

$$\sum_{j \geq 0} \Big\{ x^{j+s-2}(1-x^2)(j+s)(j+s-1)a_j - 2(j+s)a_j x^{j+s}$$

$$+ n(n+1)a_j x^{j+s}\Big\} = 0$$

$$\sum_{j \geq 0} \Big\{(j+s)(j+s-1)x^{j-2+s} a_j + \big[-(j+s)(j+s-1) - 2(j+s) + n(n+1)\big]$$

$$\times x^{j+s} a_j \Big\} = 0$$

Re-index: $m = j-2 \quad j = m+2$

$$\sum_{m=-2}^{\infty} (m+s+2)(m+s+1)x^{m+s} a_{m+2} = s(s-1)a_0 x^{s-2}$$

$$+ (s+1)s\, x^{s-1} a_1 + \sum_{j \geq 0} (j+s+2)(j+s+1) x^{j+s} a_{j+2}$$

Combine: $s(s-1)a_0 x^{s-2} = 0$ Indicial eqn.

$s(s+1)a_1 x^{s-1} = 0$

and

$$a_{j+2} = \frac{[(j+s)(j+s+1) - n(n+1)]}{(j+s+2)(j+s+1)} a_j \qquad \text{recursion relation}$$

2 roots $s=0$ & $s=1$.

$\boxed{s=0}$ $$a_{j+2} = \frac{-n(n+1)+(j+1)j}{(j+2)(j+1)}\, a_j$$

if $a_0 \neq 0$ & $a_1 = 0$; we get an even series.

$$y_e(x) = a_0\left(1 - \frac{n(n+1)}{2}x^2 + \cdots\right)$$

$\boxed{s=1}$ $a_1 = 0$ is required but even though we get a_j for even j, we have x^{1+j} giving odd powers.

$$a_{j+2} = \frac{[(j+1)(j+2) - (n+1)n]}{(\cancel{j}+3)(j+2)}\, a_j$$

$-n^2 - n + 2 = -(n^2+n-2)$
$= -(n-1)(n+2)$

$$y_o(x) = a_1\left(x + \frac{2-n(n+1)}{3!}x^3 + \cdots\right)$$
$$= a_1\left(x - \frac{(n+2)(n-1)}{3!}x^3 + \cdots\right)$$

Of course, if $n \in \mathbb{N}_{even}$ then $y_e(x)$ is a polynomial, and if $n \in \mathbb{N}_{odd}$ then $y_o(x)$ is a polynomial. Each has the parity of n.

8.3.2. $\langle u, v\rangle_w = \int_{-\infty}^{\infty} \bar{u}(x) v(x) e^{-x^2}\, dx$.

$(Lv)(x) = v'' - 2xv'$. Show, provided u & v decay at infinity, that $\langle Lu, v\rangle_w = \langle u, Lv\rangle_w$.

$$\langle u, Lv\rangle = \int_{-\infty}^{\infty} \bar{u}(x) v''(x) e^{-x^2}\, dx - 2\int \bar{u}(x)\, x v'(x) e^{-x^2}\, dx$$

(5-3)

1st integral:

$$\int_{\mathbb{R}} \bar{u}(x)\, v''(x) e^{-x^2} dx = -\int_{\mathbb{R}} \left(e^{-x^2}\bar{u}(x)\right)' v'(x) \quad \text{provided} \quad \lim_{M\to\infty} \left[e^{-x^2} v'(x)\bar{u}(x)\right]\Big|_{-M}^{M} = 0$$

$$= +\int_{\mathbb{R}} \left[\frac{d}{dx}\left(e^{-x^2}u(x)\right)'\right]^{c.c.} v(x)\, dx$$

$$\frac{d}{dx}\left(e^{-x^2}u(x)\right) = \left(-2x u(x) + u'(x)\right)e^{-x^2}$$

$$\frac{d}{dx}\left(e^{-x^2}u(x)\right)' = (-2x)(-2xu + u')e^{-x^2}$$

$$+ \left(-2u - 2xu' + u''\right)e^{-x^2}$$

$$= \left(u'' + 4x^2 u - 4xu' - 2u\right)e^{-x^2}$$

2nd integral is

$$\int_{\mathbb{R}} (\bar{u}\, x e^{-x^2}) v' = -\int_{\mathbb{R}} \left[\left(u x e^{-x^2}\right)'\right]^{cc} v$$

provided $\lim_{M\to\infty} \bar{u}(x) e^{-x^2} x \bar{v}(x)\Big|_{-M}^{M} = 0$

$$= -\int_{\mathbb{R}} \overline{\left(xu' + u - 2x^2 u\right)} e^{-x^2} v$$

Add:

$$\langle u, Lv\rangle = \int_{\mathbb{R}} \overline{(u'' - 2xu')}\, v\, e^{-x^2} dx = \langle Lu, v\rangle$$

8.3.4. Laguerre's ODE $xy'' + (1-x)y' + ny = 0$.

Weight:

$$w(x) = \frac{1}{x}\exp\left\{\int\left(\frac{1-x}{x}\right)dx\right\} = \frac{1}{x}\exp\{\log x - x\}$$

$$= e^{-x} \quad \text{on } [0,\infty).$$

$wLy = w(x)xy'' + w(x)(1-x)y'$ is hermitian

$$= \left(\frac{d}{dx}(xe^{-x})\frac{d}{dx}y\right)(x)$$

$$(xe^{-x})' = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

$$= (1-x)w(x).$$

$x_0 = 0$: $\lim_{x\to 0} x\left(\frac{1-x}{x}\right) = 1$ $\quad \lim_{x\to 0} x^2\left(\frac{n}{x}\right) = 0$

so $x_0 = 0$ reg sing pt.

$y(x) = \sum_j a_j x^{j+s}$

$$\sum_{j=0}^{\infty}\left\{(j+s)(j+s-1)a_j x^{j+s-1} + (j+s)a_j x^{j+s-1} - (j+s)a_j x^{j+s} + na_j x^{j+s}\right\} = 0$$

Re-index:

$$\sum_{j=0}^{\infty}(j+s)^2 a_j x^{j+s-1} = \sum_{m=-1}^{\infty}(m+s+1)^2 a_{m+1} x^{m+s}$$

$m = j-1$
$j = m+1$

$$= s^2 a_0 x^{s-1} + \sum_{j=0}^{\infty}(j+s+1)^2 a_{j+1} x^{j+s}$$

Indicial eqn. $s^2 = 0$

Recursion rel. $a_{j+1} = \left[\frac{j+s-n}{(j+s+1)^2}\right]a_j$

Take $s=0$. $\boxed{a_{j+1} = \frac{j-n}{(j+1)^2}a_j}$

Since this is a one-term recursion relation we get only one soln. a_0 determines a_1.

If n is a positive integer the series terminates & gives a poly of degree n.

$$a_1 = -n a_0$$
$$a_2 = \frac{1-n}{2^2} a_1 = (-1)^2 \frac{n(n-1)}{2^2} a_0$$
$$a_3 = \frac{2-n}{3^2} a_2 = (-1)^3 \frac{n(n-1)(n-2)}{2^2 \cdot 3^2} a_0$$

etc. $a_{n+1} = 0$ if $n \in \mathbb{N}$.

5.2.1. Gram Schmidt. on $[0,1]$ Start with LI set $\{1, x, x^2, \ldots\}$ construct ON set. Call $f_n(x) = x^n$, $n = 0, 1, 2, \ldots$

$u_1 = f_0(x) = 1$ normalized

$$u_2: \quad \tilde{f}_1(x) = f_1(x) - \left(\int_0^1 1 \cdot x\right) 1 = x - \left(\frac{x^2}{2}\Big|_0^1\right) \cdot 1$$

$$= x - \tfrac{1}{2} \qquad u_2 = \frac{(x - \frac{1}{2})}{\left[\int_0^1 (x-\frac{1}{2})^2\right]^{\frac{1}{2}}} = \sqrt{12}(x - \tfrac{1}{2}) \quad \text{normalise.}$$

$$u_3: \quad \tilde{f}_2(x) = f_2(x) - \left(\int_0^1 u_2 f_2\right) u_2 - \left(\int_0^1 u_1 f_2\right) u_1$$

$$\int_0^1 u_2 f_2 = \int_0^1 \sqrt{12}(x - \tfrac{1}{2}) x^2 \, dx = \left(\frac{1}{4} - \frac{1}{6}\right)\sqrt{12} = \frac{1}{\sqrt{12}}$$

$$\int_0^1 u_1 f_2 = \int_0^1 1 \cdot x^2 \, dx = \frac{1}{3}$$

$$\tilde{f}_2(x) = x^2 - \left(x - \tfrac{1}{2}\right) - \tfrac{1}{3}$$

$$= x^2 - x + \tfrac{1}{6} \qquad u_3 = \frac{x^2 - x + \frac{1}{6}}{\left[\int_0^1 (x^2 - x + \frac{1}{6})^2\right]^{\frac{1}{2}}}$$

Normalize: $u_3(x) = \sqrt{5}(6x^2 - 6x + 1)$

So 3 ON vectors are $u_1(x) = 1$, $u_2(x) = \sqrt{12}(x - \frac{1}{2})$ and $u_3(x) = \sqrt{5}(6x^2 - 6x + 1)$

The problem asks for 3 fnc. - mutually orthogonal - so $P_j(1) = 1$. These are just constant multiples of the u_j: $P_1(1) = 1$ $P_2(1) = 2x - 1$

$P_3(1) = 6x^2 - 6x + 1$

This preserves <u>orthogonality</u> but the P_j's are no longer L^2 - normalized