MA/PHY 506

PS6-1

Solutions to PS 6

1. 8.2.10 Arfken: $L_\alpha y = (1-x^2)y'' - (2\alpha+1)xy'$ on $[-1,1]$

Eigenvalue problem: $Ly = -n(n+2\alpha)y$
without weight.

Now $(1-x^2)' = -2x$ this isn't equal to $-(2\alpha+1)x$
so compute a weight:

$$\int -\frac{(2\alpha+1)x}{(1-x^2)}\,dx = \frac{1}{2}(2\alpha+1)\log(1-x^2)$$

$$w(x) = \frac{1}{1-x^2} e^{\frac{1}{2}(2\alpha+1)\log(1-x^2)} = \frac{1}{1-x^2}\cdot(1-x^2)^{\alpha+\frac{1}{2}}$$

$$= (1-x^2)^{\alpha-\frac{1}{2}}$$

Whence

$$\mathcal{L}_\alpha = w L_\alpha = (1-x^2)^{\alpha+\frac{1}{2}}\frac{d^2}{dx^2} - (2\alpha+1)(1-x^2)^{\alpha-\frac{1}{2}}x$$

$$= \frac{d}{dx}(1-x^2)^{\alpha+\frac{1}{2}}\frac{d}{dx}$$

and the ev problem with weight:

a) $$\mathcal{L}_\alpha y = \frac{d}{dx}(1-x^2)^{\alpha+\frac{1}{2}}y' = -n(n+2\alpha)w_\alpha y$$

b) This is the Sturm-Liouville problem on $L^2([-1,1])$
Since $(1-x^2)^{\alpha+\frac{1}{2}}$ vanishes at $x=\pm1$, no other
BC are needed $(\alpha > -\frac{1}{2})$.
If we solve $L_\alpha y = -(n+2\alpha)n\,y$

PS6-2

$$a_{j+2} = \frac{(j+s)(j+s+2\alpha) - n(n+2\alpha)}{(j+s+2)(j+s+1)} a_j$$

we get:

$s(s-1) = 0$ indicial eqn.

$s=0$: $$a_{j+2} = \frac{j(j+2\alpha) - n(n+2\alpha)}{(j+2)(j+1)} a_j$$

$a_0 \neq 0$

$a_1 = 0$. We get a polynomial soln. of degree n $C_n^{(\alpha)}(x)$. when n is even.

$s=1$ $a_0 \neq 0$

$a_1 = 0$ $$a_{j+2} = \frac{(j+1)(j+1+2\alpha) - n(n+2\alpha)}{(n+3)(n+2)} a_j$$

since j is even, $(j+1)$ is odd so we get a polynomial soln when n is odd.

So for each $n = 0, 1, 2, \ldots$ we get a polynomial soln. These will be orthogonal with the IP

$$\int_{-1}^{1} w_\alpha(x)\, C_n^{(\alpha)}(x)\, C_m^{(\alpha)}(x)\, dx = 0, \; n \neq m.$$

2. $f \in L^2([a,b])$ & $\{\varphi_j\}_{\infty}$ ON basis.

$$f = \sum_{j=1}^{\infty} \langle \varphi_j, f \rangle \varphi_j = \sum_{j=1}^{\infty} c_j(f)\varphi_j$$

let $S_N(x) = \sum_{j=1}^{N} c_j \varphi_j$ (these are any coefficients c_j)

$$MS_N(f, \{c_j\}) = \int_a^b |f(x) - S_N(x)|^2 dx$$

PS6-3

$$MS_N(f) = \|f\|^2 - \int_a^b \overline{f(x)} S_N(x) - \int \overline{S_N(x)} f(x) + \sum_{j=1}^{N} c_j^2$$

Assume everything is real.

$$= \|f\|^2 - 2\sum_{j=1}^{N} c_j(f)\, c_j + \sum_{j=1}^{N} c_j^2$$

Minimize $\frac{\partial MS_N(f, \{c\})}{\partial c_k} = -2c_k(f) + 2c_k$

Set equal to zero: $c_k = c_k(f)$

Is this a minimum? $\frac{\partial^2 MS_N(f, \{c\})}{\partial c_k^2} = 2 > 0$

Yes - fnc is concave up.

- When the minimum occurs when $c_k = c_k(f)$ for all k.

3. $y'' = x(x-2\pi)$ $[0, \pi]$

S-L: $-y'' = \lambda y$ DBC ef $\varphi_j(x) = \sqrt{\frac{2}{\pi}} \sin nx$

$\lambda_j = n^2$ $n = 1, 2, 3, \ldots$

$$h(x) = x(x-2\pi)$$

$$= \sum c_j(h)\varphi_j(x)$$

$$y(x) = \sum c_\ell(y)\varphi_\ell(x)$$

Note the minus sign

Substitute: $y'' = -\sum_{\ell=1}^{\infty} \ell^2 c_\ell(y)\varphi_\ell(x) = \sum_{\ell=1}^{\infty} c_\ell(h)\varphi_\ell(x)$

(PS6.4)

Solve:

$$c_\ell(y) = -\sum_{\ell=1}^{\infty} \frac{c_\ell(h)}{\ell^2}\varphi_\ell(x)$$

$$c_\ell(h) = \int_0^\pi x(x-2\pi)\varphi_\ell(x)\,dx$$

Computations

$$\int_0^\pi \sin nx \sin mx = \begin{cases} \int_0^\pi \sin^2 nx\,dx & n=m \\ \int_0^\pi \left[\cos(n+m)x - \cos(n-m)x\right] & n\neq m \end{cases}$$

$n \neq m$

$$\int_0^\pi \cos \cancel{kx} = \frac{1}{k}\sin kx\Big]_0^\pi = 0.$$

any $k \neq 0$.

$$\int_0^\pi \sin^2 nx\,dx = \int_0^\pi \frac{1}{2}(1-\cos 2nx)\,dx$$

$$= \frac{\pi}{2}$$

so the normalized ef are $\varphi_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$

To compute $c_\ell(h)$; integrate by parts:

$$A = \int_0^\pi x\sin(\ell x)\,dx = \frac{1}{\ell}\left(\frac{\cancel{\sin}\,\ell x}{\ell} - x\cos \ell x\right)\Bigg|_0^\pi = -\frac{\pi}{\ell}(-1)^\ell$$

$$B = \int_0^\pi x^2\sin(\ell x)\,dx = \frac{-x^2}{\ell}\cos \ell x\Big|_0^\pi + \frac{2}{\ell^2}\left(\frac{\cos \ell x}{\ell} + x\sin \ell x\right)\Bigg|_0^\pi$$

PS6-5

$$B = -\frac{\pi^2}{\ell}(-1)^{\ell} + \frac{2}{\ell^2}\left(\frac{(-1)^{\ell}}{\ell}\right) - \frac{2}{\ell^3}$$

$$= \frac{\pi^2}{\ell}(-1)^{\ell+1} + \frac{2}{\ell^3}\left((-1)^{\ell}-1\right)$$

So...

$$c_\ell(h) = \int_0^{\pi} x(x-2\pi)\sqrt{\frac{2}{\pi}}\sin(\ell x)\,dx$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\frac{\pi^2}{\ell}(-1)^{\ell+1} + \frac{2}{\ell^3}\left((-1)^{\ell}-1\right) + \frac{2\pi^2}{\ell}(-1)^{\ell}\right]$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left[\frac{\pi^2}{\ell^2}(-1)^{\ell} + \frac{2}{\ell^3}\left((-1)^{\ell}-1\right)\right]$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\begin{cases}\left(\frac{\pi}{\ell}\right)^2 & \ell \text{ even}\\ -\left(\frac{\pi}{\ell}\right)^2\left(1+\frac{4}{\ell\pi^2}\right) & \ell \text{ odd}\end{cases}$$

PS6.6

5.1.1 $f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x)$ $\quad \{\varphi_n\}$ ONB of a Hilbert sp.

The coef. $\{a_n\}$ are unique: if $f = \sum b_n \varphi_n$ is another expansion then

$$0 = \sum (a_n - b_n)\varphi_n$$

Take the IP with φ_k

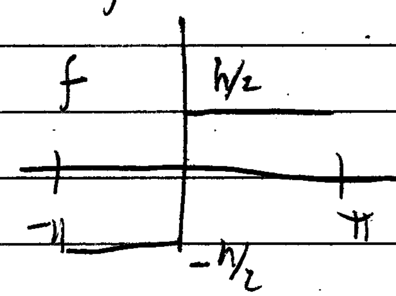
$$0 = \sum (a_n - b_n)\langle \varphi_n \varphi_k \rangle = \sum (a_n - b_n)\delta_{nk} = a_k - b_k$$

so $a_k = b_k$. In fact, we have:

$$a_n = \langle \varphi_k, f \rangle$$

so the coef. is determined by f & the ONB.

5.1.5 a) $f(x) = \begin{cases} h/2 & 0 < x < \pi \\ -h/2 & -\pi < x < 0 \end{cases}$



Expand in a sine series

$$f(x) = \frac{2h}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} \quad (f \text{ is real})$$

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \left(\frac{2h}{\pi}\right)^2 \sum_{n,m=0}^{\infty} \frac{1}{(2n+1)(2m+1)} \int_{-\pi}^{\pi} \sin(2n+1)x \sin(2m+1)x$$

$$= \frac{4h^2}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Use: $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ so

$$\int_{-\pi}^{\pi} \cos 2(n-m)x = \begin{cases} 2\pi & m = n \\ 0 & m \neq n \end{cases}$$