MA/PHY 506 Fall 2012
Midterm Exam - 100 Points
19 October 2012

INSTRUCTIONS: PLEASE WORK ALL FOUR PROBLEMS BELOW. NO BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	25	
2	25	
3	25	
4	25	
TOTAL	100	

Problem 1. (25 points.) The Laguerre ODE arises in the eigenvalue problem for the hydrogen atom. It is the ODE:

$$xy'' + (1-x)y' + ny = 0,$$

for a positive integer n.

i. Classify the finite singular points, if any.

ii. Solve the ODE by the Frobenius method about $x_0 = 0$. Indicate clearly the indicial equation and the recursion formula.

iii. Show that this equation has a polynomial solution $L_n(x)$ and write these polynomials for $n = 0, 1, 2$.

i.) $x_0 = 0$. $\quad y'' + \underbrace{\left(\frac{1-x}{x}\right)}_{P(x)}y' + \underbrace{\frac{n}{x}}_{Q(x)}y = 0 \qquad \lim_{x\to 0} x\cdot\left(\frac{1-x}{x}\right) = 1$ finite ✓

$\lim_{x\to 0} x^2\left(\frac{n}{x}\right) = 0$ finite ✓

so $x_0 = 0$ is a regular singular point.

ii.) let $y(x) = \sum_{k\geq 0} a_k x^{k+s}$ so $y'(x) = \sum_{k\geq 0}(k+s)a_k x^{k+s-1}$

$y''(x) = \sum_{k\geq 0}(k+s)(k+s-1)a_k x^{k+s-2}$

Substitute into the ODE:

$$\underbrace{\sum_{k=0}^{\infty}(k+s)^2 a_k x^{k+s-1}}+\sum_{k=0}^{\infty}(n-k-s)a_k x^{k+s} = 0$$

$$s^2 a_0 x^{s-1} + \sum_{k=0}^{\infty}(k+1+s)^2 a_{k+1}x^{k+s} + \sum_{k=0}^{\infty}(n-k-s)a_k x^{k+s} = 0$$

↓ indicial eqn $s^2 = 0$ so $s = 0$.

Recursion relation: $\boxed{a_{k+1} = \frac{(-n+k)}{(k+1)^2}a_k,\ k = 0, 1, \ldots}$

Note that we get only one soln. this way.

iii.) Polynomials: since $a_{k+1} = \frac{(k-n)}{(k+1)^2}a_k$ as soon as $k = n$

$a_{n+1} = 0$ so the soln. is $y(x) = \sum_{k=0}^{n} a_k x^k$

$\underline{n=0}$ $\quad L_n(x) = 1$ (we get $y(x) = a_0$)

$\underline{n=1}$ $\quad L_1(x) = a_0 + (-1)a_0 x) = a_0(1-x)$

$\underline{n=2}$ $\quad L_2(x) = a_0 + (-2x)a_0 + \frac{1}{2}x^2 a_0 = \left(1 - 2x + \frac{1}{2}x^2\right)a_0$.

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Problem 2. (25 points.) Find the most general solution to the Euler nonhomogeneous ODE:

$$x^2y'' + xy' = 1,$$

for $x > 0$. If you use the Green's function, make sure you divide through by x^2. Integrate by parts to do some of the integrals.

Homog. ODE: $x^2y'' + xy' = 0$ let $u = y'$

$y_1(x) = 1$ clearly a soln.

For the second: $x^2u' + xu = 0$ separate.

$\frac{du}{u} = -\frac{dx}{x} \Rightarrow \log u = -\log x + C$

$u(x) = Ce^{-\log x} = C/x = y'(x)$

$y(x) = C\log x + D$ (note that D is $Dy_1(x)$).

So take $y_2(x) = \log x$. $\{y_1, y_2\}$ are LI for $x > 0$

$W(y_1,y_2)(x) = \begin{vmatrix} 1 & \log x \\ 0 & \frac{1}{x} \end{vmatrix} = \frac{1}{x} > 0, \ x > 0.$

Green's fnc: $G_L(x,s) = \frac{y_2(x)y_1(s) - y_1(x)y_2(s)}{W(y_1,y_2)(s)}$

$= s\log x - s\log s.$

Then $y'' + \frac{1}{x}y' = \frac{1}{x^2}$ means $f(s) = \frac{1}{s^2}$.

$y_p(x) = \int_1^x G(x,s)f(s)\,ds = \int_1^x \frac{\log x}{s}\,ds - \int_1^x \frac{\log s}{s}\,ds$

$= (\log x)^2 - \frac{1}{2}(\log x)^2$ since $\int \frac{\log s}{s} = \frac{1}{2}(\log s)^2 + C$

$= \frac{1}{2}(\log x)^2.$

Check: $y_p' = \frac{\log x}{x}$ $y_p'' = \frac{1}{x^2} - \frac{\log x}{x^2}$ so $x^2y_p'' + xy_p'$

$= 1 - \log x + \log x = 1$ ✓

$$\boxed{y_{gen}(x) = A\cdot 1 + B\cdot\log x + \frac{1}{2}(\log x)^2}$$

Problem 3. (25 points.)

i. Find the most general solution of the ODE: $xy' + y = xe^{-x^2}$.

ii. Find the solution to the initial value problem: $x^2yy' + yy' = x$ and $y(0) = 0$.

(i) $y' + \frac{1}{x}y = e^{-x^2}$ $\quad p(x) = \frac{1}{x}$ $\quad$ Integrating factor $\mu(x) = e^{\int^x p(t)dt} = x$

$\frac{d}{dx}(\mu y) = \mu q = xe^{-x^2}$ so $(\mu y)(x) = \int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + C$

$\boxed{y(x) = -\frac{1}{2x}e^{-x^2} + \frac{C}{x}}$ $\quad$ check: $y'(x) = e^{-x^2} - \frac{C}{x^2} + \frac{1}{2x^2}e^{-x^2}$

then $y' + \frac{1}{x}y = e^{-x^2} - \frac{C}{x^2} + \frac{1}{2x^2}e^{-x^2} - \frac{1}{2x^2}e^{-x^2} + \frac{C}{x^2} = e^{-x^2}$ ✓

(ii) $(1+x^2)yy' = x$ $\quad$ Separate. $y\,dy = \frac{x}{1+x^2}dx$

$\frac{1}{2}y^2 = \frac{1}{2}\log(1+x^2) + C$

$y^2(x) = \log(1+x^2) + C$.

check $2yy' = \frac{2x}{1+x^2}$ ✓

Initial Cond: $y(0) = 0 = \log 1 + C \Rightarrow C = 0$.

$\boxed{y^2(x) = \log(1+x^2)}$

Problem 4. (25 points.) Find the most general solution to the nonhomogeneous ODE:

$$y'' - 4y' + 4y = e^x.$$

Be sure to clear describe each step of your calculation.

Homog. ODE: $r^2 - 4r + 4 = 0 = (r-2)^2$ $\quad y_1(x) = e^{2x}$

$y_2(x) = xe^{2x}$

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(2x+1) \end{vmatrix} = e^{4x}$$

Variation of Parameters.

$$C_1(x) = -\int_0^x \frac{y_2(s) f(s)}{W(s)}\, ds = -\int_0^x \frac{se^{2s} \cdot e^s}{e^{4s}} = -\int_0^x se^{-s}\, ds$$

$$\int se^{-s}\, ds = -se^{-s}\Big|_0^x - \left(\int_0^x (-e^{-s})\, ds\right) = -xe^{-x} - e^{-x}$$

$$= -(1+x)e^{-x} + d$$

$C_1(x) = (1+x)e^{-x}$

$$C_2(x) = \int_0^x \frac{y_1(s) f(s)}{W(s)}\, ds = \int_0^x \frac{e^{2s} e^s}{e^{4s}}\, ds = \int_0^x e^{-s}\, ds$$

$$= -e^{-x} + 1$$

$$y_p(x) = +(1+x)e^{-x} \cdot e^{2x} + (-e^{-x})xe^{2x} = e^x$$

check: $y_p'' - 4y' + 4y = (1-4+4)e^x = e^x$. ✓

$$\boxed{y_g(x) = Ae^{2x} + Bxe^{2x} + e^x}$$