PHY/MA 506 PS9-1

1. $f = \sum d_j \phi_j \qquad \|f\|^2 = \sum |d_j|^2$ and with $d_j = \langle \phi_j, f \rangle$

$S_N = \sum_{j=1}^{N} c_j \phi_j$ for any c_j - we want to find the c_j that minimize:

- $MS_N(f, \{c_j\}) = \int_a^b |f - S_N|^2 \, dx$

$$= \|f\|^2 - 2\mathrm{Re}\langle f, S_N \rangle + \sum_{j=1}^{N} c_j^2$$

$$\langle f, S_N \rangle = \sum_{j=1}^{N} c_j \langle f, \phi_j \rangle = \sum_{j=1}^{N} c_j \overline{d_j}$$

so $MS_N(f, \{c_j\}) = \|f\|^2 - 2\sum_{j=1}^{N} c_j \langle f, \phi_j \rangle + \sum_{j=1}^{N} c_j^2$

Minimize: $\frac{d}{dc_k} MS_N(f, \{c_j\}) = -2\langle f, \phi_k \rangle + 2c_k = 0$

$$\Rightarrow c_k = \langle f, \phi_k \rangle \quad \text{(assumed real)}$$

Note that $\frac{d^2}{dc_k^2} MS_N(f, \{c_j\}) = 2 > 0$

meaning $MS_N(f, \{c_j\})$ is concave up as a fnc of c_k so we have a minimum when $c_k = d_k = \langle f, \phi_k \rangle$.

2. $Ly = \lambda y = -y''$ on $[0, \pi]$ with DBC.

eigenfncs: $\varphi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx) \qquad \int_0^\pi \varphi_j^2 \, dx = \pi/2$

ev's: $\lambda_j = j^2 \quad j = 1, 2, 3, \ldots$

PS9-2

Nonhomog ODE: $y'' = x(x-2\pi) = h(x)$

$$y(x) = \sum c_j \varphi_j(x) \qquad c_j \text{ unknown}$$
$$h(x) = \sum d_j \varphi_j(x) \qquad d_j \text{ known}$$

$$-c_j \lambda_j \varphi_j = d_j \varphi_j \qquad \text{(note minus sign)}$$

$$c_j = -\frac{d_j}{\lambda_j}, \quad \lambda_j \geq 1$$

$$d_j = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin(jx)\, h(x)\, dx = \sqrt{\frac{2}{\pi}} \int_0^\pi (x^2 - 2\pi x) \sin(jx)\, dx$$

Integrals:

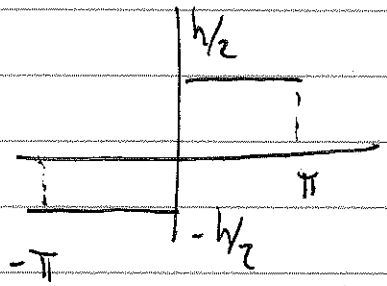
$$\int_0^\pi x \sin(jx)\, dx = \frac{1}{j}\left(\frac{\sin jx}{j} - x\cos jx\right)\Big|_0^\pi = -\frac{\pi(-1)^j}{j}$$

$$\int_0^\pi x^2 \sin(jx)\, dx = -\frac{x^2}{j}\cos(jx)\Big|_0^\pi$$
$$+ \frac{2}{j^2}\left(\frac{\cos jx}{j} + x\sin jx\right)\Big|_0^\pi$$
$$= -\frac{\pi^2}{j}(-1)^j + \frac{2}{j^2}\left(\frac{(-1)^j - 1}{j}\right) = \frac{\pi^2}{j}(-1)^{j+1} + \frac{2}{j^3}\left((-1)^j - 1\right)$$

so $$d_j = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[\frac{\pi^2}{j}(-1)^{j+1} + \frac{2}{j^3}\left((-1)^j - 1\right) + \frac{2\pi^2}{j}(-1)^{j+1}\right]$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[\frac{\pi^2}{j}(-1)^j + \frac{2}{j^3}\left((-1)^j - 1\right)\right]$$

3. Square Wave



period 2π

Odd fnc so $a_k = 0$ and $b_k = \frac{2}{\pi}\int_0^{\pi} \frac{h}{2}\sin(kx)\,dx$

$$b_k = \left(\frac{2}{\pi}\right)\left(\frac{h}{2}\right)\int_0^{\pi}\sin(kx)\,dx = -\frac{h}{\pi}\left(\frac{\cos kx}{k}\right)\Big|_0^{\pi}$$

$$= -\frac{h}{k\pi}\left(\cos k\pi - 1\right)$$

$$= \begin{cases} 0 & k \text{ even} \\ \frac{2h}{k\pi} & k \text{ odd} \end{cases}$$

So

$$f(x) = \sum_{k=1}^{\infty} \frac{2h}{(2k-1)\pi}\sin(2k-1)x = \frac{2h}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{(2k-1)}$$

Note that the series is 0 at $x=0$

this equals $\frac{f(0^+)+f(0^-)}{2} = \left(\frac{h}{2}-\frac{h}{2}\right)\frac{1}{2} = 0$

Similarly at $x = \pm\pi$, the series is 0

which is the average

$$\frac{f(\pi^-)+f(-\pi^+)}{2} = 0.$$