MA/PHY 506

PS6-1

Solutions to PS 6

1. $y'' + 2y' + y = 4e^{-x}$

Homog. ODE: $r^2 + 2r + 1 = 0$

$$r_{\pm} = \frac{-2 \pm [4-4]^{\frac{1}{2}}}{2} = -1$$

$y_1(x) = e^{-x}$

$y_2(x) = xe^{-x}$

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x}(1-x) \end{vmatrix} = e^{-2x}[1-x+x]$$

$$= e^{-2x} > 0.$$

The 2 solns are independant.

Green's fnc $$G(x,w) = \frac{y_2(x)y_1(w) - y_2(w)y_1(x)}{W(y_1,y_2)(w)}$$

$$= \frac{xe^{-x}e^{-w} - we^{-w}e^{-x}}{e^{-2w}} = xe^{w-x} - we^{w-x}$$

$$= e^{w-x}(x-w)$$

Particular soln: $$y_p(x) = \int_{x_0}^{x} G(x,w)h(w)\,dw$$

$$= 4\int_{x_0}^{x} (x-w)e^{w-x}e^{-w}\,dw$$

$$= 4\int_{x_0}^{x} (x-w)e^{-x}\,dw$$

$$= 4\left(\frac{x^2}{2}e^{-x} - x^2e^{-x}\right) - 2x_0^2e^{-x} + 4x_0^2xe^{-x}$$

$$= 2x^2e^{-x} + (y_{homog})(x)$$

PS6-2

Check: $y_p(x) = 2x^2e^{-x}$

$y_p'(x) = +4xe^{-x} - 2x^2e^{-x}$

$y_p''(x) = +4e^{-x} - 4xe^{-x} - 4xe^{-x} + 2x^2e^{-x}$

$= -8xe^{-x} + 4e^{-x} + 2x^2e^{-x}$

Substitute

$y'' + 2y' + y = (+2x^2 - 4x^2 + 2x^2)e^{-x}$

$(8x - 8x)e^{-x}$

$(+4)e^{-x} = 4e^{-x}$ ✓

Alternate method:

$c_1'(x) = \frac{-y_2'(x)h(x)}{W(x)} = \frac{-xe^{-x}\,4e^{-x}}{e^{-2x}} = -4x$

$c_2'(x) = \frac{y_1(x)h(x)}{W(x)} = \frac{e^{-x}\,4e^{-x}}{e^{-2x}} = 4$

$c_1(x) = 2x^2 \qquad c_2(x) = 4x$

$y_p(x) = -2x^2e^{-x} + 4x^2e^{-x} = 2x^2e^{-x}$ ✓

General Soln:

$y_g(x) = c_1e^{-x} + c_2xe^{-x} + 2x^2e^{-x}$,

PS6-3

2. $f_0(t) = 1$ $\qquad w_0(t) = \frac{1}{\sqrt{2}}$ $\qquad \|w_0\| = 1$

$f_1(t) = t$ $\qquad$ on $[-1,1]$

$f_2(t) = t^2$

$w_1(t)$: $\qquad u_1(t) = f_1(t) - \langle w_0, f_1 \rangle w_0$

$$= t - \left(\int_{-1}^{1} \frac{1}{\sqrt{2}} t\,dt\right) \frac{1}{\sqrt{2}}$$

$$= t - \frac{1}{4} \cdot t^2 \Big|_{-1}^{1} = t$$

$$\|u_1\|^2 = \int_{-1}^{1} t^2\,dt = \frac{1}{3} \cdot 2 = \frac{2}{3}$$

$$w_1(t) = \sqrt{\frac{3}{2}}\, t$$

$w_2(t)$: $\qquad u_2(t) = f_2(t) - \langle w_0, f_2 \rangle w_0 - \langle w_1, f_2 \rangle w_1$

$f_2(t) = t^2$ even $\qquad w_1(t) = t\sqrt{\frac{3}{2}}$ odd so $\langle w_1, f_2 \rangle = 0$

$$\langle w_0, f_2 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} t^2\,dt = \frac{1}{\sqrt{2}} \left(\frac{2}{3}\right) = \frac{\sqrt{2}}{3}$$

$w_2(t) = \sqrt{\frac{5}{8}}\left(3t^2 - 1\right)$ $\qquad u_2(t) = t^2 - \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = t^2 - \frac{1}{3}$

$$\int_{-1}^{1} u_2^2(t)\,dt = \int_{-1}^{1} \left(t^4 - \frac{2}{3} t^2 + \frac{1}{9}\right) dt$$

$$= \frac{2}{5} - \frac{\overset{18}{\cancel{4}}}{9} + \frac{2}{9} = \frac{18 \cdot 20 + 10}{45} = \frac{8}{45}$$

PS6-4

3. $v_1 = (2, 1, -1)$
$v_2 = (3, 2, 1)$
$v_3 = (1, 0, -3)$

$2v_1 - v_2 = v_3$ and $v_1 \neq \lambda v_2$ for any $\lambda \in \mathbb{R}$
so $\{v_1, v_2\}$ are linearly indep.
The dim of the subspace of $\mathbb{R}^3$ spanned by $\{v_1, v_2, v_3\}$ is 2.

Another way to see this: $\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & 0 & -3 \end{pmatrix}$ has <u>rank 2</u>

by row reduction.

In general, n vectors $\{v_j\}_{j=1}^{n}$ are LI if the determinant of the $n \times n$ matrix $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$

is <u>not</u> zero. (so the matrix has full rank n).

PS6-5

4. Bessel's inequality.

Let $V_k = \sum_{j=1}^{k} \langle v_j, v \rangle v_j$

$$\|V_k\|^2 = \sum_{j=1}^{k} |\langle v_j, v \rangle|^2$$

If $\dim V = k$, then $V_k = V$ & $\|V\|^2 = \sum_{j=1}^{k} |\langle v_j, v \rangle|^2$

If $\dim V > k$, then let $\{w_{k+1}, \dots, w_N\}$ be the other ON vectors so $\{v_1, \dots, v_k, w_{k+1}, \dots, w_N\}$ is an ONB.

$$\|V\|^2 = \sum_{j=1}^{k} |\langle v_j, v \rangle|^2 + \sum_{m=1}^{N-k} |\langle w_{k+m}, v \rangle|^2$$

$$\geq \sum_{j=1}^{k} |\langle v_j, v \rangle|^2.$$

(PS6-6)

5. $f \in L^2_{\mathbb{R}}([a,b])$ $\quad \{\phi_j\}$ ONB $\quad f = \sum \lambda_j(f)\phi_j$

$\lambda_j(f) = \langle \phi_j, f \rangle$

$$S_N(x) = \sum_{j=1}^{N} c_j \phi_j(x)$$

$$MS_N(f, \{c_j\}) = \int_a^b |f(x) - S_N(x)|^2 \, dx$$

Find $\{c_j\}$ minimizing this.

$$MS_N(f, \{c_j\}) = \langle f - S_N, f - S_N \rangle$$

$$= \|f\|^2 - 2\langle S_N, f \rangle + \|S_N\|^2$$

$$= \sum_{j=1}^{\infty} |\lambda_j(f)|^2 - 2\sum_{j=1}^{N} c_j \lambda_j(f) + \sum_{j=1}^{N} |c_j|^2$$

$$\frac{\partial}{\partial c_k} MS_N(f, \{c_j\}) = -2\lambda_k(f) + 2c_k = 0$$

$$c_k = \lambda_k(f) \qquad k = 1, \dots, N$$

$$\frac{\partial^2}{\partial c_k^2} MS_N(f, \{c_j\}) = 2 > 0$$ so each

critical pt. is a minimum.

You can also write:

$$\|f - S_N\|^2 = \sum_{j=N+1}^{\infty} |\lambda_j(f)|^2 + \sum_{j=1}^{N} (\lambda_j(f) - c_j)^2 \geq 0$$

minimized when $\lambda_j(f) = c_j$.