1. \[ y'' + 2y' + y = 4e^{-x} \]

Homog. ODE: \[ r^2 + 2r + 1 = 0 \]
\[ r = -2 \pm \frac{[4-4]^\frac{1}{2}}{2} = -1 \]

\[ y_1(x) = e^{-x} \]
\[ y_2(x) = xe^{-x} \]

\[ W(y_1,y_2)(x) = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x}(1-x) \end{vmatrix} = e^{-2x}(1-x+x) = e^{-2x} \]

The 2 solns are independent.

Green's func \[ G(x,w) = \frac{y_2(x)y_1(w) - y_1(w)y_2(x)}{W(y_1,y_2)(w)} \]

\[ = \frac{xe^{-x}e^{-w} - we^{-w}e^{-x}}{e^{-2w}} = xe^{w-x} - we^{-w}e^{-x} \]
\[ = e^{w-x}(x-w) \]

Particular soln: \[ y_p(x) = \int_x^y G(x,w)h(w)\,dw \]
\[ = y \int_x^y (x-w)e^{w-x}e^{-w}\,dw \]
\[ = y \int_x^y (x-w)e^{-x}\,dw \]
\[ = 4y \left( \frac{x^2e^{-x} - x^2e^{-x}}{2} + \frac{y}{x}xe^{-x} \right) \]
\[ = 2xe^{-x} + (y_{homog})(x) \]
Check: \( y_p(x) = 2x^2e^{-x} \)

\[
y_p'(x) = 4xe^{-x} - 2xe^{-x}
\]

\[
y_p''(x) = 4e^{-x} - 4xe^{-x} = 4xe^{-x} + 2x^2e^{-x}
\]

\[
= -8xe^{-x} + 4e^{-x} + 2x^2e^{-x}
\]

Substitute

\[
y'' + 2y' + y = \frac{(2x^2 - 4xe^{-x} + 2xe^{2x})e^{-x}}{(8x - 8x)} = 4e^{-x}
\]

Alternate method:

\[
c_1(x) = -\frac{y_1(x)h(x)}{W(x)} = \frac{-xe^{-x}Ye^{-x}}{e^{-2x}} = -4x
\]

\[
c_2(x) = \frac{y_2(x)h(x)}{W(x)} = \frac{e^{-x}Ye^{-x}}{e^{-2x}} = 4
\]

\[
c_1(x) = 2x^2 \quad c_2(x) = 4x
\]

\[
y_p(x) = -2x^2e^{-x} + 4xe^{-x} = 2x^2e^{-x}
\]

General Soln:

\[
y_g(x) = c_1e^{-x} + c_2xe^{-x} + 2x^2e^{-x},
\]
2. \( f_0(t) = 1 \)
   \( f_1(t) = t \) on \([-1, 1]\)
   \( f_2(t) = t^2 \)

\[ W_0(t) = \frac{1}{\sqrt{2}} \quad \|W_0\| = 1 \]

\[ w_1(t) = \frac{1}{\sqrt{2}} \left( \frac{1}{t^2} \int t^2 dt \right) \frac{1}{\sqrt{2}} = t \]

\[ \|w_1\|^2 = \int_{-1}^{1} t^2 dt = \frac{1}{3} \cdot 2 = \frac{2}{3} \]

\[ w_1(t) = \sqrt{\frac{2}{3}} t \]

\[ w_2(t) = f_2(t) - \langle w_1, f_2 \rangle w_0 - \langle w_1, f_1 \rangle w_1 = \frac{t^2}{\sqrt{2}} - \frac{1}{\sqrt{2}} \int_{-1}^{1} t^2 dt = \frac{1}{\sqrt{2}} \left( \frac{2}{3} \right) = \frac{\sqrt{2}}{3} \]

\[ \langle w_2, f_2 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} t^2 dt = \frac{1}{\sqrt{2}} \left( \frac{2}{3} \right) = \frac{\sqrt{2}}{3} \]

\[ w_2(t) = \sqrt{\frac{5}{8}} \left( 3t^2 - 1 \right) \]

\[ \|w_2\|^2 = \int_{-1}^{1} t^2 dt = \frac{1}{3} \cdot 2 = \frac{2}{3} \]

\[ \int_{-1}^{1} w_2(t) dt = \int_{-1}^{1} \left( t^2 - \frac{2}{3} t + \frac{1}{9} \right) dt \]

\[ = \left[ \frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{9} \right]_{-1}^{1} = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{18.20 + 10}{45} = \frac{28}{45} \]
3. \[ \mathbf{v}_1 = (2, 1, -1) \]
\[ \mathbf{v}_2 = (3, 2, 1) \]
\[ \mathbf{v}_3 = (1, 0, 3) \]

\[ 2\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3 \text{ and } \mathbf{v}_1 \neq \lambda \mathbf{v}_2 \text{ for any } \lambda \in \mathbb{R} \]

so \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) are linearly independent.

The dimension of the subspace of \( \mathbb{R}^3 \) spanned by \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is 3.

Another way to see this:

\( \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \) has rank 2

by row reduction.

In general, \( n \) vectors \( \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \) are LI if the determinant of the \( n \times n \) matrix

\( \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \)

is not zero (so the matrix has full rank \( n \)).
Bessel's inequality

Let \( V_k = \sum_{j=1}^{K} \langle v_j, v \rangle v_j \)

\[ \| V_k \| = \sum_{j=1}^{K} | \langle v_j, v \rangle | \]

If \( \dim V = k \), then \( V_k = V \) and \( \| V \| = \sum_{j=1}^{K} | \langle v_j, v \rangle | \)

If \( \dim V > k \), then let \( \{ w_{k+1}, \ldots, w_N \} \) be the other ON vectors so \( \{ v_1, \ldots, v_k, w_{k+1}, \ldots, w_N \} \) is an ONB.

\[ \| V \| = \sum_{j=1}^{K} | \langle v_j, v \rangle | + \sum_{m=1}^{N-k} | \langle w_m, v \rangle | \]

\[ \geq \sum_{j=1}^{K} | \langle v_j, v \rangle | \]
5. \( f \in L^2([a,b]) \) \{ \phi_j \} ONB \[ f = \sum_{j=1}^{\infty} \lambda_j(f) \phi_j \]

\[ S_N(x) = \sum_{j=1}^{N} c_j \phi_j(x) \]

\[ M_{S_N}(f, \{ c_j \}) = \int_{a}^{b} \left| f(x) - S_N(x) \right|^2 \, dx \]

Find \( \{ c_j \} \) minimizing this

\[ M_{S_N}(f, \{ c_j \}) = \langle f - S_N, f - S_N \rangle \]

\[ = \| f \|^2 - 2 \langle S_N, f \rangle + \| S_N \|^2 \]

\[ = \sum_{j=1}^{\infty} |a_j(f)|^2 - 2 \sum_{j=1}^{\infty} c_j a_j(f) + \sum_{j=1}^{\infty} |c_j|^2 \]

\[ \frac{\partial}{\partial c_k} M_{S_N}(f, \{ c_j \}) = -2 a_k(f) + \sum_{j=1}^{\infty} c_j = 0 \]

\[ c_k = a_k(f) \quad k = 1, \ldots, N \]

\[ \frac{\partial^2}{\partial c_k^2} M_{S_N}(f, \{ c_j \}) = 2 > 0 \text{ so each critical pt. is a minimum} \]

You can also write:

\[ \| f - S_N \|^2 = \sum_{j=1}^{\infty} |a_j(f)|^2 + \sum_{j=1}^{N} (a_j(f) - c_j)^2 \geq 0 \]

Minimized when \( \lambda_j(f) = c_j \).