PHY/MA 506

(t1-1)

Solutions to Test 1

1. $t^2 y'' - 2y = 3t^2 - 1, \quad t > 0$

Standard form: $y'' - \frac{2}{t^2} y = 3 - 1/t^2, \quad t > 0$

1) Homog ODE $t^2 y'' - 2y = 0$. Guess $y(t) = t^\alpha$

$\alpha(\alpha - 1) - 2 = 0 \qquad \alpha^2 - \alpha - 2 = 0$

$r_1 = 2, \ r_2 = -1 \qquad \frac{1 \pm [1+8]^{1/2}}{2} = \frac{1}{2} \pm \frac{3}{2}$

$y_1(t) = t^2, \quad y_2(t) = t^{-1}$

2) $W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -3 \neq 0 \quad$ L.I.

Basis for soln. space; soln. space is 2 dimensional and spanned by $\{y_1, y_2\}$.

3) $G(t,s) = \dfrac{y_2(t)\, y_1(s) - y_2(s)\, y_1(t)}{W(y_1, y_2)(s)}$

$= (t^{-1}s^2 - s^{-1}t^2)/(-3) = \frac{1}{3}\left[s^{-1}t^2 - t^{-1}s^2\right]$

Particular soln:

$y_p(t) = \int_1^t G(t,s)\, h(s)\, ds = \frac{t^2}{3}\int_1^t \frac{1}{s}\left(3 - \frac{1}{s^2}\right) ds - \frac{t^{-1}}{3}\int_1^t s^2\left(3 - \frac{1}{s^2}\right)$

Note: Guarantees $y_p(1) = 0$

$= \frac{t^2}{3}\left(3\log t + \frac{1}{2}\left(\frac{1}{t^2} - 1\right)\right) - \frac{t^{-1}}{3}\left((t^3 - 1) - (t - 1)\right)$

General soln

$\left\{\begin{array}{l} y_p(t) = t^2 \log t - \frac{1}{2}t^2 + \frac{1}{2} \\ y_p(1) = 0 \end{array}\right\}, \quad y_g(t) = A_1 t^2 + A_2 t^{-1} + y_p(t)$

(11-2)

2. $xy'' + (1-x)y' + \lambda y = 0$

$\lim_{x\to 0}\left(\frac{1-x}{x}\right)x = 1$

$\lim_{x\to 0}\left(\frac{\lambda}{x}\right)x^2 = 0$

$x_0 = 0$ reg sing. pt.

$y(x) = \sum_{j=0}^{\infty} x^{s+j} a_j$

$y'(x) = \sum_{j=0}^{\infty} (s+j) x^{s+j-1} a_j$

$y''(x) = \sum_{j=0}^{\infty} (s+j)(s+j-1) x^{s+j-2} a_j$

$$\sum_{j=0}^{\infty} \left[(s+j)(s+j-1)a_j + (s+j)a_j\right] x^{s+j-1}$$

$$+ \sum_{j=0}^{\infty} \left[\lambda - (s+j)\right] a_j x^{s+j} = 0$$

Re-index 1^{st} sum: $m = j-1 \quad j \geq 1 \Rightarrow m \geq 0$

1^{st} Sum $= \sum_{m=0}^{\infty} (s+m+1)^2 a_{m+1} x^{s+m} + s^2 a_0 x^{s-1}$

Indicial eqn $s^2 = 0 \quad s = 0$, so $a_0 \neq 0$

Recursion: $\sum_{j=0}^{\infty} \left[(j+1)^2 a_{j+1} + (\lambda - j) a_j\right] = 0$

$$\boxed{a_{j+1} = \frac{j-\lambda}{(j+1)^2} a_j}$$

First few terms: $a_0 \qquad a_1 = -\lambda a_0$

$$a_2 = \frac{1-\lambda}{2^2} a_1 = \frac{\lambda(\lambda-1)}{2^2} a_0$$

$$a_3 = \frac{2-\lambda}{3^2} a_2 = -\frac{\lambda(\lambda-1)(\lambda-2)}{3^2 2^2} a_0$$

so

$$\boxed{y(x) = a_0\left[1 - \lambda x + \frac{\lambda(\lambda-1)}{2^2} x^2 - \frac{\lambda(\lambda-1)(\lambda-2)}{2^2 3^2} x^3 + \dots\right]}$$

<u>Wronskian</u> Abel's formula $P(x) = \left(\frac{1-x}{x}\right)$ (standard form)

$$W(y_1, y_2)(x) = W_0 e^{-\int^x P(s)\,ds}$$
$$= W_0 e^{-\log x + x} = W_0 \frac{e^x}{x}, \quad W_0 \neq 1 \text{ if } 2 \text{ solns are LI.}$$

<u>2nd Soln</u> $y_2(x) = y_1(x) \int^x \frac{1}{y_1(t)^2} e^{-\int^t P(s)\,ds}\,dt$

$$= y_1(x) \int^x \frac{1}{y_1(t)^2} \left(\frac{e^t}{t}\right) dt$$

$y_1(x) = \sum_{j=0}^{\infty} a_j x^j$ as above & is regular at $x=0$.

$$y_2(x) = a_0\left\{1 - \lambda x + \frac{\lambda(\lambda-1)x^2}{2^2} + \dots\right\} \int^x \frac{1}{1-2\lambda t + O(t)^2} \frac{e^t}{t}\,dt$$

$$= a_0\left\{1 - \lambda x + O(x^2)\right\}\left\{\int^x \frac{1}{1-2\lambda t + O(t)^2}\left\{\frac{1 + t + t^2/2 + \dots}{t}\right\}\right.$$

$$= a_0\left\{1 - \lambda x + O(x^2)\right\}\left\{\int^x (1 + 2\lambda t)\left(\frac{1}{t} + 1 + \frac{t}{2} + \dots\right) dt\right.$$

(t1-4)

so for x close to 0:

$$y_2(x) = a_0 \{ \log x + b_1 x \log x + \cdots \}$$

$y_2(x) \sim \log x$ as $x \to 0^+$.

$\lambda \in \mathbb{N}$ Series for $y_1(x)$ terminates.

$\lambda = m \in \mathbb{N}$; $a_{m+1} = 0$ $a_m = \frac{-1}{m^2} a_{m-1}, \ldots$

Laguerre polynomials. (normalised)

$$L_0(x) = 1$$
$$L_1(x) = 1 - x$$
$$L_2(x) = \frac{x^2 - 4x + 2}{2!}$$
$$L_3(x) = \frac{-x^3 + 9x^2 - 18x + 6}{9!}$$

(t1-5)

3. $$\left(y\cos x + 2xe^y\right) + \left(\sin x + x^2e^y - 1\right)\frac{dy}{dx} = 0$$

$P(x,y) = y\cos x + 2xe^y$

$\frac{\partial P}{\partial y} = \cos x + 2xe^y$

Exact

$Q(x,y) = \sin x + x^2e^y - 1$

$\frac{\partial Q}{\partial x} = \cos x + 2xe^y$

exact: Find $\varphi(x,y)$ so $d\varphi(x,y) = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = 0$

$\frac{\partial\varphi}{\partial x} = P(x,y)$

$\varphi(x,y) = \int P(x,y)dx + h(y) = y\sin x + x^2e^y + h(y)$

$\frac{\partial\varphi}{\partial y} = Q(x,y)$

$\varphi(x,y) = \int Q(x,y)dy + g(x) = y\sin x + x^2e^y - y + g(x)$

Compare: $y\sin x + x^2e^y - y = \text{const.}$

#1-6

4.

$$ty'(t) + 2y(t) = t(t-1) \qquad t>0$$

$$y'(t) + \frac{2}{t}y(t) = (t-1)$$

Integrating factor: $\mu = e^{\int^t p(s)ds}$

$$= e^{\int^t \frac{2}{s}ds} = e^{2\log t}$$

$$= t^2$$

$\frac{d}{dt}(\mu y)' = \mu q$ so $(\mu y)(t) = \int_1^t s^2(s-1)\,ds$

Integral: $\int_1^t (s^3 - s^2)\,ds = \left(\frac{t^4}{4} - \frac{1}{4}\right) + \left(-\frac{t^3}{3} + \frac{1}{3}\right)$

$$= \frac{t^4}{4} - \frac{t^3}{3} + \frac{1}{12}$$

so

$$y(t) = t^{-2}\left[\frac{t^4}{4} - \frac{t^3}{3} + \frac{1}{12}\right] = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{12t^2}$$

Note

$$y(1) = 0$$

and

$$\left.\begin{aligned} y'(t) &= \frac{1}{2}t - \frac{1}{3} - \frac{1}{6t^3} \\ \frac{2y(t)}{t} &= \frac{t}{2} - \frac{2}{3} + \frac{1}{6t^3} \end{aligned}\right\} y' + \frac{2}{t}y = t-1$$