MA/PHY 506 Fall 2018
Final Exam - 100 Points
12 December 2018

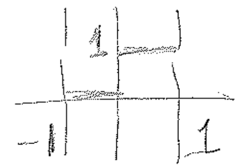
INSTRUCTIONS: PLEASE WORK ALL FIVE PROBLEMS BELOW. NO BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	100	

Problem 1. (20 points.) Compute the Fourier series of the function f on $[-1,1]$ given by

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \in [-1,0] \end{cases}$$



Period = 2.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\pi x + b_k \sin k\pi x \right)$$

$$a_k = \int_{-1}^{1} f(x) \cos k\pi x \, dx \qquad k = 0, 1, 2, \ldots$$

$$b_k = \int_{-1}^{1} f(x) \sin k\pi x \, dx \qquad k = 1, 2, \ldots$$

Compute $k \neq 0$

$$\cdot\ a_k = \int_0^1 \cos k\pi x \, dx = \left. \frac{\sin k\pi x}{k\pi} \right|_0^1 = \frac{\sin k\pi}{k\pi} = 0$$

$$a_0 = 1$$

$$\cdot\ b_k = \int_0^1 \sin k\pi x \, dx = \left. -\frac{\cos k\pi x}{k\pi} \right|_0^1 = \frac{1 - \cos k\pi}{k\pi}$$

$$= \frac{1 - (-1)^k}{k\pi} = \begin{cases} 0 & k \text{ even} \\ \frac{2}{k\pi} & k \text{ odd} \end{cases}$$

so

$$\boxed{f(x) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{2}{2j-1} \sin[(2j-1)\pi x]}$$

One way to see this is that $f(x) - \frac{1}{2}$ is an odd fnc.

Problem 2. (20 points.) Suppose that a linear transformation A has a matrix representation of the form

$$[A] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

relative to the standard orthonormal basis $\{e_1, e_2\}$ for R^2.

i. Find the eigenvalues and eigenvectors of A. Identify clearly the characteristic polynomial for A.

ii. Find an orthonormal basis of R^2 consisting of eigenvectors of A.

iii. Construct the matrix S that diagonalizes A and carry-out the diagonalization.

i.) $P_A(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda$

3 A is self-adjoint. $= \lambda(\lambda - 2)$

eigenvalues $\sigma(A) = \{0, 2\}$, the zeros of $P_A(\lambda)$.

eigenvectors:

3 $\underline{\lambda = 0}$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \iff a + b = 0$. so $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = u_1$

3 $\underline{\lambda = 2}$ $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \iff a = b$ so $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = u_2$

9 $\underline{\text{Check}}$. $\|u_1\| = 1 = \|u_2\|$ $(u_1, u_2) = 0$ so orthonormal.

$Au_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\, u_1$ ✓

$Au_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2u_2$ ✓

Each ev is simple.

4 ii) $\{u_1, u_2\}$ is an ONB of $\mathbb{R}^2$.

iii) Find S orthogonal so $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = S^{-1}AS$ (checked below)

$S = (u_1 \mid u_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ 3 so $S\hat{e}_1 = u_1$ & $S\hat{e}_2 = u_2$

7 $S^* = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $S^*S = 1$ $\Big\}$ $S^* = S^{-1}$. orthogonal.

Finally: $S^{-1}AS = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} S = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ ✓

Diagonal for of A

Problem 3. (20 points.) Consider the damped harmonic oscillator ODE:

$$(Lx)(t) = x''(t) + 2\lambda x'(t) + \omega_0^2 x(t) = 0.$$

Assume $0 < \lambda < \omega_0$ corresponding to weak dampening.

1. Find a basis of the solution space of this ODE. Make sure you verify that the solutions $\{x_1(t), x_2(t)\}$ form a basis.
2. Write the Green's function $G_L(t, s)$ for this differential operator L.
3. Suppose the oscillator receives a kick at time $t = \tau_0$. Find the corresponding particular solution to the nonhomogeneous ODE with driving term $f_{\tau_0}(t) = \delta(t - \tau_0)$. The delta function at τ_0 models kicking the oscillator at $t = \tau_0$. The nonhomogeneous ODE is:

$$(Lx)(t) = x''(t) + 2\lambda x'(t) + \omega_0^2 x(t) = f_{\tau_0}(t).$$

i, $r^2 + 2\lambda r + \omega_0^2 = 0$

roots $r_\pm = \left(-2\lambda \pm [4\lambda^2 - 4\omega_0^2]^{\frac{1}{2}}\right)\frac{1}{2} = -\lambda \pm (\lambda^2 - \omega_0^2]^{\frac{1}{2}}$

since $\omega_0 > \lambda > 0$ $\lambda^2 - \omega_0^2 < 0$ so

$r_\pm = -\lambda \pm [\omega_0^2 - \lambda^2]^{\frac{1}{2}} i$

let $\omega = \sqrt{\omega_0^2 - \lambda^2}$

2 solns. $x_1(t) = e^{-\lambda t}\cos([\omega_0^2 - \lambda^2]^{\frac{1}{2}} t)$

4 $x_2(t) = e^{-\lambda t}\sin([\omega_0^2 - \lambda^2]^{\frac{1}{2}} t)$

They are LI and a basis of the soln. space (2 dim.) of $Lx = 0$.

LI follows from $W(x_1, x_2)(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix}$

$= \begin{vmatrix} e^{-\lambda t} c\omega t & e^{-\lambda t}\sin\omega t \\ e^{-\lambda t}(-\lambda c\omega t - \omega s\omega t) & e^{-\lambda t}(-\lambda s\omega t + \omega c\omega t) \end{vmatrix}$

$= e^{-2\lambda t}\left[(-\lambda s\omega t\, c\omega t + \omega c^2\omega t) - (-\lambda s\omega t\, c\omega t - \omega s^2\omega t)\right]$

3 $= \omega e^{-2\lambda t} \neq 0$ implying LI.

6 ii. $G(t,s) = \dfrac{x_1(s)x_2(t) - x_1(t)x_2(s)}{W(s)} = \dfrac{e^{-\lambda(s+t)}[\cos s\omega t - \cos s\omega t]}{\omega e^{-2\lambda s}}$

$$\boxed{G_L(t,s) = \frac{1}{\omega} e^{-\lambda(t-s)} \sin\omega(t-s)}$$

iii) $$y_p(t) = \int_0^t G_L(t,s)\, h(s)\, ds$$

$$= \int_0^t \frac{1}{\omega} e^{-\lambda(t-s)} \sin\omega(t-s)\, \delta(s-\tau_0)\, ds$$

$$= \begin{cases} 0 & 0 < t < \tau_0 \\ \frac{1}{\omega} e^{-\lambda(t-\tau)} \sin\omega(t-\tau) & \tau_0 < t \end{cases}$$

Problem 4. (20 points.)

1. Compute the Fourier transform of the function

$$f(x) = \begin{cases} 1 & x \in [-1, 1] \\ 0 & |x| > 1 \end{cases}$$

2. Using the integral representation of the delta function,

$$\delta(x-y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(x-y)k}\, dk,$$

compute the integral

$$\int_{-\infty}^{\infty} e^{-ikx} \cos(\pi x)\, dx.$$

1.

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ikx}}{-ik} \right) \Bigg|_{-1}^{1} = \frac{i}{k\sqrt{2\pi}} \left(e^{-ik} - e^{ik} \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin k}{k} \right)$$

2.

$$\int_{-\infty}^{\infty} \cos(\pi x) e^{-ikx}\, dx = \frac{1}{2} \left[\int_{-\infty}^{\infty} e^{-i(k-\pi)x}\, dx + \int_{-\infty}^{\infty} e^{-i(k+\pi)x}\, dx \right]$$

$$= \pi \left[\delta(k-\pi) + \delta(k+\pi) \right]$$

Problem 5. (20 points.) We know the that Legendre operator

$$L = \frac{d}{dx}(1-x^2)\frac{d}{dx}$$

is self-adjoint on $L^2([-1,1])$ and the eigenfunctions $P_\ell(x)$, the Legendre polynomials, satisfy

$$(LP_\ell)(x) = \ell(\ell+1)P_\ell(x), \quad \ell = 0, 1, 2, \ldots.$$

The eigenfunctions $\{P_n(x) \mid n = 0, 1, 2, \ldots\}$ form a basis of $L^2([-1,1])$ and satisfy

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = \frac{2}{2n+1}\delta_{nm}.$$

1. Write the eigenfunction expansion of any $h \in L^2([-1,1])$. Make sure to identify the coefficients in the expansion.

2. Using the eigenfunction expansion, write a series solution of the nonhomogeneous ODE:
$$(Lf)(x) = h(x),$$
for any $h \in L^2([-1,1])$.

1. $(f, Lg) = \int_{-1}^{1} f(x) \frac{d}{dx}(1-x^2)g'(x)\,dx = (Lf, g)$ so self-adjoint.

$L P_\ell(x) = \ell(\ell+1) P_\ell(x)$

Normalize $\tilde{P}_n(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x)$

$\{\tilde{P}_n(x)\}$ ONB of $L^2([-1,1])$

$$\boxed{h(x) = \sum_{n=0}^{\infty} h_n \tilde{P}_n(x) \text{ where } h_n = (\tilde{P}_n, h) = \sqrt{\frac{2n+1}{2}} \int_{-1}^{1} P_n(x) h(x)\,dx}$$

2. Expand the soln. $f(x) = \sum_{n=0}^{\infty} f_n \tilde{P}_n(x)$

Substitute: $Lf = \sum f_n n(n+1) \tilde{P}_n = \sum_{n=0}^{\infty} h_n \tilde{P}_n$

Take IP

$(\tilde{P}_\ell, Lf) = \sum_n f_n n(n+1)(\tilde{P}_\ell, \tilde{P}_n) = \ell(\ell+1) f_\ell = h_\ell$

$$\boxed{f_\ell = \frac{h_\ell}{\ell(\ell+1)},\ \ell \neq 0, \text{ and we require } h_0 = 0}$$