PHY/MA 506 PS2-1

Solutions to PS # 2

1. $y'' + 2y' + 2y = 0$

$r^2 + 2r + 2 = 0$

$r_\pm = \dfrac{-2 \pm [4-8]^{1/2}}{2} = -1 \pm i$

2 complex solns $\quad y_\pm(x) = e^{-x \pm ix}$

$= e^{-x}(\cos x \pm i \sin x)$

These are complex conjugates.

2 real solns. $\quad y_1(x) = e^{-x}\cos x$

$y_2(x) = e^{-x}\sin x$

These are independent! If there is a constant

$\lambda \neq 0$ so $\quad y_1(x) = \lambda y_2(x)$ for all x

then $\quad \cos x = \sin x$

Take $x = 0$: $\quad 1 = 0 \quad$ absurd!

Basis for the real soln. space $\{y_1, y_2\}$

General soln:

$$y(x) = Ay_1(x) + By_2(x)$$

Initial cond: $\quad y(0) = 1 \quad \& \quad y'(0) = 2$

To find the unique soln write

$y(0) = A = 1$

$y'(x) = -e^{-x}(A\cos x + B\sin x)$

$+ e^{-x}(-A\sin x + B\cos x)$

$y'(0) = -A + B = -1 + B = 2 \quad$ so $B = 3$.

$$\boxed{y(x) = e^{-x}\cos x + 3e^{-x}\sin x}$$

$y'(0) = -1 + 3 = 2$ ✓

PS2-2

2. Solve $x''(t) + \omega_0 x(t) = 0$ by Power Series

$x(t) = \sum_{j \geq 0} a_j t^j \qquad x'(t) = \sum_{j \geq 0} j a_j t^{j-1}, \quad \omega_0 > 0$

and $x''(t) = \sum j(j-1) a_j t^{j-2}$, $x_0 = 0$ is a reg. pt.

Substitute:

$$(*) \quad \sum_{j=0}^{\infty} j(j-1) a_j t^{j-2} + \omega_0 a_j t^j = 0$$

Re-index the 1st sum:

$m = j - 2$

$j = m + 2 \quad m = 0, 1, \ldots$

$$\sum_{j=0}^{\infty} j(j-1) a_j t^{j-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m$$

(note $j=0$ & $j=1$ give no contribution)

$(*)$ becomes $\displaystyle\sum_{j=0}^{\infty} \left((j+2)(j+1) a_{j+2} + \omega_0 a_j \right) t^j = 0$

$\Leftrightarrow$

$$\boxed{a_{j+2} = -\frac{\omega_0}{(j+2)(j+1)} a_j}$$

Recursion relation

$a_0 \neq 0, \; a_1 = 0$:

$$a_2 = \frac{-\omega_0}{2 \cdot 1} a_0$$

$$a_4 = \frac{-\omega_0}{4 \cdot 3} a_2 = \frac{(-1)^2 (\omega_0^2)}{4 \cdot 3 \cdot 2 \cdot 1} a_0$$

PS2-3

$$a_6 = \frac{-w_0}{6\cdot 5} a_4 = \frac{(-1)^3 w_0^3}{6!} a_0$$

even terms $2p$:

$$a_{2p} = \frac{(-1)^p w_0^p}{(2p)!} a_0$$

1st Soln:

$$y_1(t) = a_0 \left(\sum_{p=0}^{\infty} \frac{(-1)^p (w_0^{\frac{1}{2}} t)^{2p}}{(2p)!} \right)$$

$$= a_0 \left(1 - \frac{1}{2} w_0 t^2 + \frac{1}{4!} w_0^2 t^4 - \dots \right)$$

$$= a_0 \cos(w_0^{\frac{1}{2}} t)$$

2nd Soln:

$$a_3 = \frac{-w_0}{3\cdot 2} a_1$$

$$a_5 = \frac{-w_0}{5\cdot 4} a_3 = \frac{(-w_0)^2}{5!} a_1$$

$$a_7 = \frac{-w_0}{7\cdot 6} a_5 = \frac{(-w_0)^3}{7!} a_1$$

odd terms $2p+1$

$$a_{2p+1} = \frac{(-w_0)^p}{(2p+1)!} a_1$$

$$y_2(t) = a_1 \left(\sum_{p=0}^{\infty} \frac{(-w_0)^p}{(2p+1)!} t^{2p+1} \right)$$

$$= a_1 \left(t - \frac{w_0}{3!} t^3 + \frac{w_0^2}{5!} t^5 - \frac{w_0^3}{7!} t^7 + \dots \right)$$

PS 2-4

Relate to $\sin(\omega_0^{\frac{1}{2}} t)$

$$y_2(t) = \left(\frac{a_1}{\omega_0^{\frac{1}{2}}}\right)\left(\omega_0^{\frac{1}{2}} t - \frac{(\omega_0^{\frac{1}{2}} t)^3}{3!} + \frac{(\omega_0^{\frac{1}{2}} t)^5}{5!} \cdots \right)$$
$$= \left(\frac{a_1}{\omega_0^{\frac{1}{2}}}\right) \sin(\omega_0^{\frac{1}{2}} t)$$

General Soln:

$$y(t) = A\cos(\omega_0^{\frac{1}{2}} t) + B\sin(\omega_0^{\frac{1}{2}} t)$$

(PS 2.5)

3. $y''' - 2y'' - y' + 2y = 0$ Expect 3 LI solns. - these span the solution space

3^{rd} order linear homog ODE with constant coefficients

Guess: $y(x) = e^{rx}$

Substitute: $r^3 - 2r^2 - r + 2 = 0$

One root is $r = 2$

$r^3 - 2r^2 - r + 2 = (r-2)(r^2-1)$

3 roots $\{2, 1, -1\}$

Let: $y_1(x) = e^{2x}$ $y_2(x) = e^{x}$ $y_3(x) = e^{-x}$

General Soln: $y(x) = Ae^{2x} + Be^{x} + Ce^{-x}$

Linear Indep: (1) We can use the Wronskian

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{x} & e^{-x} \\ 2e^{2x} & e^{x} & -e^{-x} \\ 4e^{2x} & e^{x} & e^{-x} \end{vmatrix}$$

$$= e^{2x}\begin{vmatrix} e^{x} & -e^{-x} \\ e^{x} & e^{-x} \end{vmatrix} - e^{x}\begin{vmatrix} 2e^{2x} & -e^{-x} \\ 4e^{2x} & e^{-x} \end{vmatrix} + e^{-x}\begin{vmatrix} 2e^{2x} & e^{x} \\ 4e^{2x} & e^{x} \end{vmatrix}$$

$$= e^{2x}(2) - e^{x}(6e^{x}) + e^{-x}(-2e^{3x})$$

$= -4e^{2x} - 2e^{2x} = -6e^{2x}$, doesn't vanish for any x

So the 3 fncs. are LI so form a basis

PS 2-6

(2) We can do it by hand: Suppose

$$c_1 e^{2x} + c_2 e^{x} + c_3 e^{-x} = 0 \text{ for all } x$$

then

$$2c_1 e^{2x} + c_2 e^{x} - c_3 e^{-x} = 0 \text{ by differentiating}$$

and

$$4c_1 e^{2x} + c_2 e^{x} + c_3 e^{-x} = 0$$

Evaluate all at $x = 0$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 1 & 1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

Show the 3×3 matrix is invertible by computing its determinant.

$$1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$$

$$= 2 - 1(6) + 1(-2) = -6 \neq 0,$$

So the only solution is $c_1 = c_2 = c_3 = 0$

This means $\{e^{2x}, e^{x}, e^{-x}\}$ are LI