PHY/MA 506

PS3-1

Solutions to PS 3

1. Hermite: $y'' - 2xy' + 2\lambda y = 0$

a) Harmonic osc: $-\psi'' + x^2\psi = E\psi$

let $\psi(x) = e^{-x^2/2}\, y(x)$

$$\psi'(x) = -x e^{-x^2/2} y(x) + e^{-x^2/2} y'(x)$$

$$\psi''(x) = x^2 \psi(x) - \psi(x) - 2x e^{-x^2/2} y'(x) + e^{-x^2/2} y''(x)$$

Substitute:

$$-\psi'' + x^2\psi = -e^{-x^2/2}\left[x^2 y(x) - y(x) - 2xy'(x) + y''\right] + e^{-x^2/2} x^2 y = E e^{-x^2/2} y(x)$$

so

$$-y'' + 2xy' + (1-E)y = 0$$

or

$$y'' - 2xy' + 2\lambda y = 0 \quad \text{with } E = 1 + 2\lambda$$

b) Recursion relation:

We studied this already

$$a_{j+2} = \frac{2(j-\lambda)}{(j+2)(j+1)} a_j$$

To have polynomial solns, we need this to terminate. This happens if λ is a positive integer (or 0). $\lambda \in \mathbb{N} \cup \{0\}$.

In this case, the eigenvalues (bound state energies) of the quantum harmonic oscillator are:

$$E_n = (2n+1) \text{ and } \psi_n(x) = \underbrace{h_n(x) e^{-x^2/2}}_{\text{Hermite functions}}$$

PS3-2

$$\begin{cases} -\psi'' + x^2\psi = E\psi \\ \psi(x) = e^{-x^2/2}\, y(x) \\ y \text{ satisfies} \\ \qquad y'' - 2xy' + 2\lambda y = 0, \quad 2\lambda = E-1 \end{cases}$$

Polynomial soln:

$$a_{j+2} = \frac{2(j-\lambda)}{(j+2)(j+1)} a_j, \quad h_\lambda(x) = \sum_{j=0}^{\infty} a_j x^j$$

$(a_0 = 1, a_1 = 0)$

$$a_2 = \frac{-2\lambda}{2} a_0 = -\lambda a_0$$

$$a_4 = \frac{2(2-\lambda)}{4\cdot 3} a_2 = \frac{2(\lambda-2)\lambda}{4\cdot 3} a_0$$

$(a_0 = 0, a_1 = 1)$

$$a_3 = \frac{2(1-\lambda)}{3\cdot 2} a_1$$

$$a_5 = \frac{2(3-\lambda)}{5\cdot 4} a_3 = \frac{2^2(1-\lambda)(3-\lambda)}{5\cdot 4\cdot 3\cdot 2} a_1$$

$\vdots$

If $\lambda \in \mathbb{N}_{\text{even}}$, the solns are even polynomial of order λ

$a_0 = 1$ $\qquad h_0(x) = 1$

$a_0 = -2$ $\qquad h_2(x) = a_0(1 - 2x^2) = -2(1-2x^2)$

$$= 4x^2 - 2$$

$a_0 = 12$ $\qquad h_4(x) = a_0\left(1 - 2x^2 + \frac{4}{3}x^4\right)$

$$= 16x^4 - 24x^2 + 12$$

etc.

P53-3

If $\lambda \in \mathbb{N}_{odd}$, the solns are odd polynomials of order λ

$a_1 = 2$ $\quad h_1(x) = a_1 x = 2x$

$a_1 = -12$ $\quad h_3(x) = a_1 x + a_3 x^3 = a_1\left(x - \frac{2}{3}x^3\right)$

$$= 8x^3 - 12x$$

$$h_5(x) = a_1\left(x - \frac{4}{3}x^3 + \frac{2^3}{30}x^5\right)$$

$a_1 = 120$ $\quad = 32x^5 - 160x^3 + 120x$

PS 3-4

2. $\mathcal{B} = \left\{ 1, x, \frac{x^2}{2}, \frac{x^3}{3!}, \dots, \frac{x^N}{N!} \right\}$ is LI for any $N \geq 1$

(7.6.3) Construct the Wronskian:

$$W = \begin{vmatrix} 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^N}{N!} \\ 0 & 1 & x & \cdots & \frac{x^{N-1}}{(N-1)!} \\ 0 & 0 & 1 & x \cdots & \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

The Wronskian $W = 1$ so the set $\mathcal{B}$ consists of LI functions.

(7.6.4) Suppose $W(y_1, y_2)(x) = y_1(x) y_2'(x) - y_1'(x) y_2(x) = 0$ then

$$\frac{d}{dx} \log y_1(x) = \frac{d}{dx} \log y_2(x)$$

Integrate

$$\log y_1(x) = \log y_2(x) + C$$

$$y_1(x) = e^{C} y_2(x)$$

so they are multiples and not LI

You can also write: $\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_2' y_1 - y_2 y_1'}{y_1^2} = 0$

so $y_2 = \lambda y_1$

PS3-5

(7.6.9) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Standard form:

$$y'' - \frac{2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

$$P(x) = \frac{-2x}{1-x^2} \qquad \int^x P(s)\,ds = -\int^x \frac{2s}{1-s^2}\,ds$$

$$= \log(1-x^2)$$

$$W(y_1,y_2)(x) = W(0)\,e^{-\int^x P(s)ds} = W(0)\,e^{-\log(1-x^2)}$$

$$= \frac{W(0)}{1-x^2},$$

for any 2 solns. of the ODE

PS3-6

3 (7.6.16) $R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = 0$

Check: $R_1(r) = r^m$ $R_1'(r) = m r^{m-1}$

$R_1''(r) = m(m-1) r^{m-2}$

Substitute:

$m(m-1) r^{m-2} + m r^{m-2} - m^2 r^{m-2}$

$= [m(m-1) + m - m^2] r^{m-2} = 0$ ✓

Second LI soln $R_2(r) = R_1(r) \int^r \frac{1}{R_1(t)^2} e^{-\int^t P(s)ds} dt$

$P(s) = \frac{1}{s}$ $\int^t \frac{ds}{s} = \log t$

$e^{-\int^t P(s)ds} = \frac{1}{t}$ so $R_2(r) = R_1(r) \int^r \frac{dt}{t^{2m+1}}$

Integral: $\int^r \frac{dt}{t^{2m+1}} = \frac{-1}{2m r^{2m}}$ up to a cnst.

Claim:

$R_2(r) = R_1(r) \frac{1}{r^{2m}} = \frac{1}{r^m}$

Check: $R_2'(r) = -m r^{-m-1}$ $R_2''(r) = m(m+1) r^{-m-2}$

Subst:

$[m(m+1) + (-m) - m^2] r^{-m-2} = 0$ ✓

LI set $\{ r^m, \frac{1}{r^m} \}$ LI so basis <u>except</u>

When $m = 0$. $R'' + \frac{1}{r}R' = 0$. $R_1(r) = 1$

$R_2(r) = \int^r \frac{dt}{t} = \log r$. $R_2(r) = \log r$

Check: $R_2'(r) = \frac{1}{r}$ $R_2''(r) = -\frac{1}{r^2}$ so $R_2'' + \frac{1}{r} R_2 = 0$ ✓

P53-7

(7.6.19) $y'' - 2xy' + 2\alpha y = 0$

Fix $\alpha = 0$. $y'' - 2xy' = 0$ so $y_1(x) = 1$ soln.

Find $y_2(x)$ using the formula:

$$P(s) = -2s$$

$$\int^t P(s)\,ds = -t^2$$

$$y_2(x) = y_1(x) \int^x \frac{dt}{y_1(t)^2} e^{-\int^t P(s)ds} = \int^x e^{t^2}\,dt.$$

The integral can't be done in closed form.

For small x behavior:

$$e^{t^2} = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots$$

$$y_2(x) \cong \int_0^x \left\{ 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots \right\} dt$$

$$\cong x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \cdots$$

This is $y_{odd}(x) = \sum_{j=0}^{\infty} a_{2j+1} x^{2j+1}$ from the power series soln.

$$y_{odd}(x) = a_1 \left[x + \frac{2(1-\alpha)}{3!} x^3 + \frac{2^2(1-\alpha)(3-\alpha)}{5!} x^5 + \cdots \right]$$

so with $\alpha = 0$ and $a_1 = 1$

$$y_{odd}(x) = x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \cdots = y_2(x)$$

PS3 : 8

Fix $\alpha = 1$ $\quad y_1(x) = x$ is a soln.

$$y_2(x) = x \int_0^x \frac{1}{t^2} e^{t^2}\, dt$$

$$\simeq x \int_0^x \left\{ \frac{1}{t^2} + 1 + \frac{t^2}{2!} + \frac{t^4}{3!} + \cdots \right\} dt$$

$$\simeq x \left\{ \frac{-1}{x} + x + \frac{x^3}{3 \cdot 2} + \frac{x^5}{5 \cdot 3!} + \cdots \right\}$$

$$= -1 + x^2 + \frac{x^4}{3 \cdot 2} + \frac{x^6}{5 \cdot 3!} + \cdots$$

Compare w/ $y_{even}(x)$:

$$y_{even}(x) = 1 + \frac{2(-\alpha)x^2}{2!} + \frac{2^2(-\alpha)(2-\alpha)x^4}{4!} + \cdots$$

$\alpha = 1$: $\quad y_{even}(x) = 1 - x^2 - \frac{x^4}{3} + \cdots$

$$= -y_2(x).$$

(p53-9)

(7.6.26) $$y'' + \frac{1-\alpha^2}{4x^2}y = 0 \quad (x>0)$$

let $y(x) = x^m$ $\quad y' = mx^{m-1}$ $\quad y'' = m(m-1)x^{m-2}$

Substitute:

$$\left[m(m-1) + \frac{1-\alpha^2}{4}\right]x^{m-2} = 0.$$

$$m^2 - m + \frac{1-\alpha^2}{4} = 0 \qquad m_{1,2} = \frac{1}{2} \pm \frac{1}{2}\left[1 - 4\left(\frac{1-\alpha^2}{4}\right)\right]^{\frac{1}{2}}$$

$$m_{1,2} = \frac{1}{2} \pm \frac{\alpha}{2} = \frac{1\pm\alpha}{2}.$$

Two Solns: $\quad y_1(x) = x^{\frac{1+\alpha}{2}}$ $\quad$ LI if $\underline{\alpha \neq 0}$

$\qquad\qquad y_2(x) = x^{\frac{1-\alpha}{2}}$,

$\underline{\alpha = 0}$ $\quad y_1(x) = x^{\frac{1}{2}}$

$$y_2(x) = x^{\frac{1}{2}} \int^x \frac{dt}{t} = x^{\frac{1}{2}} \log x$$

Compute:

$$\lim_{\alpha \to 0} \frac{y_1(x) - y_2(x)}{\alpha} = \lim_{\alpha \to 0} x^{\frac{1}{2}}\left[\frac{\left(x^{\alpha/2} - x^{-\alpha/2}\right)}{\alpha}\right]$$

Write: $$e^{\frac{\alpha}{2}\log x} - e^{-\frac{\alpha}{2}\log x} = \left(1 + \frac{\alpha}{2}\log x + O(\alpha^2)\right)$$
$$- \left(1 - \frac{\alpha}{2}\log x + O(\alpha^2)\right)$$

$$= \alpha \log x + O(\alpha^2)$$

so $$\lim_{\alpha \to 0} x^{\frac{1}{2}}\left(\frac{x^{\alpha/2} - e^{-\alpha/2}}{\alpha}\right) = x^{\frac{1}{2}} \log x.$$

p53-10

$\alpha = 0$ Verify $y_2(x) = x^{\frac{1}{2}} \log x$ is a soln. of $y'' + \frac{1}{4x^2} y = 0$.

$y_2'(x) = \frac{1}{2} x^{-\frac{1}{2}} \log x + x^{-\frac{1}{2}}$

$y_2''(x) = -\frac{1}{4} x^{-3/2} \log x + \frac{1}{2} x^{-3/2} - \frac{1}{2} x^{-3/2}$

So

$y_2'' + \frac{1}{4x^2} y = -\frac{1}{4} x^{-3/2} \log x + \frac{1}{4} x^{-3/2} \log x$

$= 0$ ✓