MA/PHY 506 PS4-1

Solutions to PS 4

7622 $(1-x^2)y'' - xy' + n^2 y = 0$

a) $n=0$, $y'' - \frac{x}{1-x^2}y' = 0$. $y_1(x) = 1$ soln.

$$y_2(x) = y_1(x)\int^x \frac{1}{y_1(t)^2} e^{-\int^t P(s)ds}$$

$$\int^t P(s)ds = \frac{1}{2}\int^t \frac{(-2s)}{1-s^2}ds = \frac{1}{2}\int^{1-t^2} \frac{du}{u} = \frac{1}{2}\log(1-t^2)$$

$u = 1-s^2$

$du = -2s\,ds$

$$e^{-\int^t P(s)ds} = e^{\log(1-t^2)^{-\frac{1}{2}}} = \frac{1}{\sqrt{1-t^2}}$$

$$y_2(x) = \int^x \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}x$$

Check: $y'(x) = \frac{1}{\sqrt{1-x^2}}$ $y''(x) = \frac{x}{(1-x^2)^{3/2}}$

so $(1-x^2)y'' - xy' = \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = 0$. ✓

b.) Solve directly: $u = y'$ $\frac{u'}{u} = \frac{x}{1-x^2}$

$\frac{d}{du}\log u = \frac{x}{1-x^2}$ or $\log u = -\frac{1}{2}\log(1-x^2)$

$u(x) = \frac{1}{\sqrt{1-x^2}}$

so $y(x) = \sin^{-1}x$.

(P3 4-2)

7.6.25

$$y_2(x) = y_1(x) f(x)$$

$$y_2' = y_1' f + y_1 f'$$

$$y_2'' = y_1'' f + 2y_1' f' + y_1 f''$$

Substitute into $y_1'' + P y_1' + Q y_1 = 0$

y_1 is one soln.

$$(y_1'' + P y_1' + Q y_1) f + 2 y_1' f' + y_1 f'' + P y_1 f' = 0$$

so

$$-\frac{1}{y_1}(2y_1' + P y_1) f' = f''$$

$v = f'$ so

$$\frac{v'}{v} = -\frac{2y_1'}{y_1} - P$$

$$\frac{d}{dx} \log v = -2 \frac{d}{dx} \log y_1 - P$$

$$\log v = \log(y_1^{-2}) - \int^x P(s) ds$$

$$v(x) = f'(x) = \frac{1}{y_1^2} e^{-\int^x P(s) ds}$$

so

$$f(x) = \int^x \frac{1}{y_1^2(t)} e^{-\int^t P(s) ds}$$

giving the same soln.

PS4-3

7.7.3

$h(x) = e^x$

$y'' + 4y = e^x \qquad y_1(x) = \cos 2x \qquad y_2(x) = \sin 2x$

$W(y_1, y_2) = \begin{vmatrix} c2x & s2x \\ -2s2x & 2c2x \end{vmatrix} = 2$

$c_1'(x) = -\dfrac{y_2(x)h(x)}{W(y_1,y_2)(x)} = -\dfrac{1}{2}\sin(2x)\,e^x$

$c_2'(x) = \dfrac{y_1(x)h(x)}{W(y_1,y_2)(x)} = \dfrac{1}{2}\cos(2x)\,e^x$

Integrals:

$$\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu)$$

$$\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu)$$

so

$c_1(x) = -\dfrac{1}{2}\displaystyle\int e^x \sin(2x)\,dx = -\dfrac{1}{2}\left(\dfrac{e^x}{5}\right)\left[\sin 2x - 2\cos 2x\right]$

$c_2(x) = \dfrac{1}{2}\displaystyle\int e^x \cos(2x)\,dx = \dfrac{1}{2}\left(\dfrac{e^x}{5}\right)\left[\cos 2x + 2\sin 2x\right]$

and

$y_p(x) = c_1(x)\,y_1(x) + c_2(x)\,y_2(x)$

$= -\dfrac{1}{10}e^x\cos 2x(\sin 2x - 2\cos 2x) + \dfrac{1}{10}e^x\sin 2x\left[\cos 2x + 2\sin 2x\right]$ ✓

$= \dfrac{1}{5}e^x(\cos^2 2x + \sin^2 2x) = \dfrac{1}{5}e^x$

Check $\left(\dfrac{1}{5} + 4\right)e^x = e^x$ ✓

PS4-4

$$y_{gen}(x) = A\cos 2x + B\sin 2x + \frac{1}{5}e^x$$

<u>Comment</u> You can guess $y_p(x) = Ae^x + B$ and find

$$y_p'' + 4y_p = Ae^x + 4Ae^x + 4B = e^x$$

$$5A = 1, \quad A = \frac{1}{5}.$$

P54-5

7.7.4 $y'' - 3y' + 2y = \sin x$

Ly = 0 $\quad r^2 - 3r + 2 = 0$, $(r-2)(r-1) = 0$ $\Big\}$ $y_1(x) = e^{2x}$ $\quad y_2(x) = e^x$

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{2x} & e^x \\ 2e^{2x} & e^x \end{vmatrix} = -e^{3x}$$

Green's fnc

$$G_2(x,y) = \frac{y_1(y)\, y_2(x) - y_1(x)\, y_2(y)}{W(y_1,y_2)(y)}$$

$$= \frac{e^{x+2y} - e^{2x+y}}{-e^{3y}} = -e^{x-y} + e^{2(x-y)}$$

$$\boxed{G_2(x,y) = e^{2(x-y)} - e^{x-y}}$$

$$y_p(x) = \int^x G_2(x,y) \sin y \, dy$$

$$= e^{2x} \int^x e^{-2y} \sin y \, dy - e^x \int^x e^{-y} \sin y \, dy$$

$$= e^{2x}\left(\frac{e^{-2x}}{5}\left[-2\sin x - \cos x\right]\right) - e^x \left(\frac{e^{-x}}{2}\left[-\sin x - \cos x\right]\right)$$

$$= -\frac{2}{5}\sin x - \frac{1}{5}\cos x + \frac{1}{2}\sin x + \frac{1}{2}\cos x$$

$$\boxed{y_p(x) = \frac{1}{10}\sin x + \frac{3}{10}\cos x}$$

PS 4-6

Check:

$$y_p'(x) = \frac{1}{10} cx - \frac{3}{10} sx$$

$$y_p''(x) = -\frac{1}{10} sx - \frac{3}{10} cx$$

$$y_p'' - 3y_p' + 2y_p = \left(\frac{-1}{10} + \frac{9}{10} + \frac{2}{10}\right) sx + \left(\frac{-3}{10} - \frac{3}{10} + \frac{6}{10}\right) cx$$

$$= \sin x \quad \checkmark$$

Comment You can also guess $y_p(x) = A\sin x + B\cos x$

$$\boxed{y_g(x) = Ae^{2x} + Be^{x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x}$$

PS4-7

$y' + xy = xy^3$ separable

$y' = (y^3 - y)x$

$$\int \frac{dy}{y(y^2-1)} = \int x\,dx$$

$$\frac{1}{y(y^2-1)} = \frac{-1}{y} + \frac{y}{y^2-1}$$

$$\int \frac{dy}{y(y^2-1)} = -\int \frac{dy}{y} + \int \frac{y}{y^2-1}\,dy = -\log y + \frac{1}{2}\log(y^2-1)$$

$$\log\left(\frac{\sqrt{y^2-1}}{y}\right) = \frac{x^2}{2} + c$$

$$\left(\frac{y^2-1}{y^2}\right)^{\frac{1}{2}} = e^{\frac{x^2}{2}+c} = Ae^{x^2/2}$$

$$y^2 - 1 = Ce^{x^2}y^2$$

$$y^2 = \frac{1}{Ce^{x^2}+1} \Rightarrow \boxed{y(x) = \frac{1}{\sqrt{Ce^{x^2}+1}}}$$

Use the Riccobi ODE $y' = py + qy^n$

In this case $y' = -xy + xy^3$

$p(x) = -x \quad q(x) = x \quad n = 3$

let $u(x) = y^{1-n}(x) = y^{-2}(x)$

$u'(x) = -2y^{-3}(x)\,y'(x)$

$-\frac{1}{2}y^3 u' = -xy + xy^3$

$y(x) = \frac{1}{\sqrt{1+Ae^{x^2}}}$

$-\frac{1}{2}u' = -xy^{-2} + x = -xu(x) + x$

$u' = 2xu(x) - 2x = 2x(u(x) - 1)$

$\log(u-1) = x^2 + c \qquad u^2 = 1 + Ae^{x^2} = y(x)^{-2}$