**MA/PHY506 Fall 2018**
**Problem Set 5**
**DUE: Monday, 15 October 2018**

1. Find the most general solution to the nonhomogeneous ODE:

$$y'' + 2y' + y = 2x.$$

2. Apply the Gram-Schmidt method to the functions $f_0(t) = 1, f_1(t) = t, f_2(t) = t^2$ on the interval $[-1, 1]$ in order to obtain an orthonormal set of functions. The inner product is

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\ dt.$$

Compare these with the first three Legendre polynomials in the text.

3. What is the dimension of the subspace of $R^3$ spanned by the vectors: $(2, 1, -1), (3, 2, 1), (1, 0, -3)$? What is a general condition so that $n$-vectors in $R^n$ are linearly independent?

4. Let $\{v_j \mid j = 1, \ldots, K\}$ be a finite orthonormal set in an inner product space $V$. For any vector $v \in V$, show that

$$\sum_{j=1}^{K} |(v, v_j)|^2 \leq \|v\|^2.$$

This is called Bessel's inequality.

5. Let $\phi_j$ be an orthonormal basis of the Hilbert space $L^2([a, b])$. For any $f \in L^2([a, b])$, the mean square (MS) error between $f$ and the finite series approximation $S_N(x) = \sum_{j=1}^{N} c_j \phi_j(x)$ is defined by

$$MS_N(f; \{c_j\}) \equiv \int_a^b |f(x) - S_N(x)|^2\ dx.$$

Assume that $f$, the coefficients $c_j$, and the basis functions $\phi_j$ are all real (for simplicity). Show that $MS(f; \{c_j\})$ is minimized with the choice $c_j = \int_a^b \phi_j(x) f(x)\ dx$, the expansion coefficients of $f$ relative to the orthonormal basis $\phi_j$.