(PS5-1)

Solutions to PS 5

Problem 1

$y'' + 2y' + y = 2x$

Homog $\quad r^2 + 2r + 1 = (r+1)^2 \quad y_1(x) = e^{-x} \quad y_2(x) = xe^{-x}$

$$G_L(x,u) = [y_1(u)\,y_2(x) - y_1(x)\,y_2(u)]\, W(y_1,y_2)(u)^{-1}$$

$$= \frac{xe^{-u-x} - ue^{-x}e^{-u}}{e^{-2u}} = (x-u)\,e^{u-x}$$

$$W(y_1,y_2)(u) = \begin{vmatrix} e^{-u} & ue^{-u} \\ -e^{-u} & e^{-u} - ue^{-u} \end{vmatrix} = e^{-2u}(1-u) + ue^{-2u} = e^{-2u}$$

$$y_p(x) = \int_0^x G_L(x,u)\,h(u)\,du = 2\int_0^x (x-u)\,e^{-(x-u)}\,u\,du$$

$$= 2xe^{-x}\int_0^x ue^{u}\,du - 2e^{-x}\int_0^x u^2e^{u}\,du$$

$$\int_0^x ue^u\,du = (u-1)e^u\Big|_0^x = (x-1)e^x + 1$$

$$\int_0^x u^2e^u\,du = u^2e^u - 2\left[(u-1)e^u\right]\Big|_0^x$$

$$= x^2e^x - 2(x-1)e^x - 2$$

$$y_p(x) = 2xe^{-x}\left[(x-1)e^x + 1\right] - 2e^{-x}\left[x^2e^x - 2(x-1)e^x - 2\right]$$

$$= 2x(x-1) + 2xe^{-x} - 2x^2 + 4(x-1) + \underbrace{4e^{-x}}_{\text{part of } y_{homog}}$$

$$= -2x + \underbrace{2xe^{-x}}_{\text{part of } y_{homog}} + 4x - 4 + 4e^{-x}$$

$$\boxed{y_p(x) = 2x - 4}$$

PS5-2

Check $y_p(x) = 2x - 4$
$y_p'(x) = 2$
$y_p''(x) = 0$

$y_p'' + 2y_p' + y_p = 4 + 2x - 4 = 2x$ ✓

General solution: $\boxed{y_g(x) = Ae^{-x} + Bxe^{-x} + 2x - 4}$

Remark Since $h(x) = 2x$ you can
guess $y_p(x) = Ax + B$
substitute and solve for A & B:
$y_p'(x) = A \quad y_p''(x) = 0$

so $y_p''(x) + 2y_p'(x) + y_p(x) = 2A + Ax + B = 2x$

$\Rightarrow A = 2, \ B = -4$

PS5: 3

Problem 2 $f_0(t) = 1 \quad f_1(t) = t \quad f_2(t) = t^2$ on $[-1,1]$

IP $(f,g) = \int_{-1}^{1} f(t)\, g(t)\, dt$ (real LVS)

$w_0(t) = \frac{1}{\sqrt{2}}$ since $\|f_0\|^2 = \int_{-1}^{1} dt = 2.$

$w_1(t)$: $u_1(t) = f_1(t) - (w_0, f_1) w_0$

$(w_0, f_1) = \frac{1}{\sqrt{2}} \int_{-1}^{1} t\, dt = 0$

$\|u_1\|^2 = \int_{-1}^{1} t^2 dt \quad \frac{2}{3}$

$w_1(t) = \sqrt{\frac{3}{2}}\, t$

$w_2(t)$ $\quad u_2(t) = f_2(t) - (w_1, f_2) w_1 - (w_0, f_2) w_0$

$(w_1, f_2) = \sqrt{\frac{3}{2}} \int_{-1}^{1} t^3 = 0$

$w_2(t) = \sqrt{\frac{5}{2}\left(\frac{3}{2}\right)}\left(t^2 - \frac{1}{3}\right)$

$= \sqrt{\frac{5}{2}}\left(\frac{3}{2}t^2 - \frac{1}{2}\right)$

$(w_0, f_2) = \frac{1}{\sqrt{2}} \int_{-1}^{1} t^2 dt = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$

$u_2(t) = t^2 - \frac{1}{3}$

$\|u_2\|^2 = \int_{-1}^{1} \left(t^4 - \frac{2}{3}t^2 + \frac{1}{9}\right) dt$

$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$

PS5-4

So we get:

$$w_0(t) = \frac{1}{\sqrt{2}}$$

$$w_1(t) = \sqrt{\frac{3}{2}}\, t$$

$$w_2(t) = \sqrt{\frac{5}{2}}\left(\frac{3}{2}t^2 - \frac{1}{2}\right)$$

Compare with the Legendre polynomials:

$$P_0(t) = 1 = \sqrt{2}\, w_0(t) \quad \text{Table 15,1}$$

$$P_1(t) = t = \sqrt{\tfrac{2}{3}}\, w_1(t)$$

$$P_2(t) = \frac{1}{2}(3t^2-1) = 2\sqrt{\frac{2}{5}}\, w_2(t)$$

so they are the same up to a constant multiple.

PS5-5

Problem 3 $V = \mathbb{R}^3$ $\quad v_1 = (2,1,-1) \quad v_2 = (3,2,1) \quad v_3 = (1,0,-3)$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & 0 & -3 \end{bmatrix}$$ matrix w/ v_j as the j^{th} row.

$$\det A = 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$
$$= 3 - 3 = 0.$$

This means the rows are not LI $\Rightarrow$ in fact,

$v_3 = 2v_1 - v_2$ so as $\{v_1, v_2\}$ are LI

the span of $\{v_1, v_2, v_3\}$ = Span$\{v_1, v_2\}$ is a

2 dim subsp. of $\mathbb{R}^3$.

(p55-6)

Problem 4 Bessel's Inequality

$\{v_j\}_{j=1}^{k}$ ON set in IPS V.

$v \in V \quad v = \sum_{j=1}^{k} (v_j, v) v_j + v^{\perp}$

where $v^{\perp} = v - \sum_{j=1}^{k} (v_j, v) v_j$

Check: $\left(v^{\perp}, \sum_{j=1}^{k} (v_j, v) v_j\right) = \sum_{j=1}^{k} (v_j, v) (v^{\perp}, v_j)$

$(v^{\perp}, v_j) = (v, v_j) - (v_j, v) = 0$.

so

$\|v\|^2 = \underbrace{\|\sum (v_j, v) v_j\|^2}_{\sum_{j=1}^{k} |(v_j, v)|^2} + \|v^{\perp}\|^2$

So: $\|v\|^2 = \sum_{j=1}^{k} |(v_j, v)|^2 + \|v^{\perp}\|^2 \geq \sum_{j=1}^{k} |(v_j, v)|^2$

PS 5-7

Problem 5

$$\|f - S_N\|^2 = \int_a^b \left| f(x) - \sum_{j=1}^{N} c_j \phi_j(x) \right|^2 dx$$

Expand $\|f - S_N\|^2 = (f - S_N, f - S_N)$

$$= \|f\|^2 - 2\sum_{j=1}^{N} c_j (\phi_j, f) + \sum_{j=1}^{N} c_j^2$$

Minimize:

$$\frac{\partial}{\partial c_\ell} MS_N = -2(\phi_\ell, f) + 2c_\ell = 0$$

(Critical point)

$$\boxed{c_\ell = (\phi_\ell, f)}$$

To check it is a minimum, use the second deriv.!

$$\frac{\partial^2}{\partial c_\ell^2} MS_N = 2 > 0 \quad \text{so a minimum.}$$