PHY/MA 506

PS6-1

Solutions to PS 6.

2.2.17 $[M_i, M_j] = iM_k$

Take the trace of both sides

$$\mathrm{Tr}\,[M_i, M_j] = \mathrm{Tr}\,(M_i M_j - M_j M_i)$$
$$= \mathrm{Tr}\, M_i M_j - \mathrm{Tr}\, M_j M_i$$
$$= \mathrm{Tr}\, M_i M_j - \mathrm{Tr}\, M_i M_j = 0$$

using the fact that $\mathrm{Tr}\,AB = \mathrm{Tr}\,BA$

So $\mathrm{Tr}\, M_k = 0$, for any k.

2.2.38 If U is unitary than $U^{-1} = U^*$. To show U^{-1} is unitary we must show $(U^{-1})^{-1} = (U^{-1})^*$

We know $(U^{-1})^{-1} = U$. The key is the procedure of taking the adjoint commutes with taking the inverse. $(A^*)^{-1} = (A^{-1})^*$.

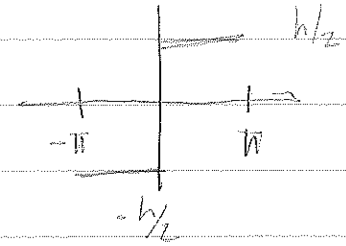
Applying this: $(U^{-1})^* = (U^*)^{-1}$ but U is unitary
$$= (U^{-1})^{-1} = U$$

Hence the statement $(U^{-1})^{-1} = (U^{-1})^*$ is true and U^{-1} is unitary.

Another way to show this:
$$U^{-1} = U^*$$
$$(U^{-1})^* = (U^*)^* = U = (U^{-1})^{-1}$$

PS6-2

5.1.5 $f(x) = \begin{cases} h/2 & 0 < x < \pi \\ -h/2 & -\pi < x < 0 \end{cases}$



Step fnc.

$$= \frac{2h}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$$

(a) $$\int_{-\pi}^{\pi} |f(x)|^2 dx = (f, f) = \left(\frac{2h}{\pi}\right)^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

The left side is $\|f\|^2 = (f, f)$

A direct calculation

$$(f, f) = \sum_{n,m=0}^{\infty} \left(\frac{2h}{\pi}\right)^2 \frac{1}{(2n+1)(2m+1)} \int_{-\pi}^{\pi} dx \sin(2n+1)x \sin(2m+1)x$$

Basic integral

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

so

$$J_{nm} = \int_{-\pi}^{\pi} dx \sin(2n+1)x \sin(2m+1)x = \frac{1}{2}\int_{-\pi}^{\pi} \Big[\cos 2(n-m)x - \cos 2[(n+m)+1]x\Big]$$

$n \neq m$ $$= \frac{1}{2}\frac{1}{2(n-m)} \sin 2(n-m)x \Big|_{-\pi}^{\pi} - \frac{1}{2}\frac{1}{2(n+m+1)} \sin(2[n+m+1]x) \Big|_{-\pi}^{\pi}$$

$n \neq m$ $J_{nm} = 0$, $n = m$ $J_{nn} = \pi$, so $J_{nm} = \pi \delta_{nm}$

so $$(f, f) = \sum_{n,m=0}^{\infty} \left(\frac{2h}{\pi}\right)^2 \frac{1}{(2n+1)(2m+1)} (\pi \delta_{nm})$$

$$= \sum_{n=0}^{\infty} \left(\frac{2h}{\pi}\right)^2 \frac{\pi}{(2n+1)^2} = \sum_{n=0}^{\infty} \frac{4h^2}{\pi} \frac{1}{(2n+1)^2}$$

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(b) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = \zeta(2)$ Riemann Zeta fnc. at $z=2$.

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1} \frac{1}{(2k)^2} = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6}$$

$2n+1$ odd integer

$$= \frac{3}{4}\frac{\pi^2}{6}$$

So

$$\|f\|^2 = \left(\frac{4h^2}{\pi}\right)\left(\frac{3}{4}\right)\left(\frac{\pi^2}{6}\right) = \frac{h^2\pi}{2}$$

This is exactly $\int_{-\pi}^{\pi} f(x)^2 dx = 2\int_0^{\pi} \left(\frac{h}{2}\right)^2 dx$

$$= \frac{h^2}{2}\pi.$$

(PS 6 - 4)

5.1.12 $\underset{\sim}{a} = a_1\hat{e}_1 + a_2\hat{e}_2$ and $\underset{\sim}{b} = b_1\hat{e}_1 + b_2\hat{e}_2$ Real LVS

Define $\langle a | b \rangle_k = a_1b_1 - a_1b_2 - a_2b_1 + k a_2b_2$
$= (a_1 - a_2)(b_1 - b_2) + (k-1) a_2b_2$

To be an inner product, we need

① $\langle \underset{\sim}{a} | \underset{\sim}{b} \rangle = \langle \underset{\sim}{b} | \underset{\sim}{a} \rangle$

~~If a~~

This is easily verified so it is symmetric

② $\langle \underset{\sim}{a} + \lambda \underset{\sim}{c} | \underset{\sim}{b} \rangle = ((a_1 + \lambda c_1 - a_2 + \lambda c_2))$

$\times (b_1 - b_2) + (k-1)(a_2 + \lambda c_2) b_2$

$= (a_1 - a_2)(b_1 - b_2) + \lambda (c_1 - c_2)(b_1 - b_2)$
$+ (k-1) a_2 b_2 + \lambda (k-1) c_2 b_2$

$= \langle \underset{\sim}{a} | \underset{\sim}{b} \rangle + \lambda \langle \underset{\sim}{c} | \underset{\sim}{b} \rangle$

so it is linear

③ $\langle \underset{\sim}{a} | \underset{\sim}{a} \rangle = (a_1 - a_2)^2 + (k-1) a_2^2$

Clearly $(k-1) \geq 0$ for $\langle \underset{\sim}{a} | \underset{\sim}{a} \rangle \geq 0$.
Further, if $a_1 = a_2 \neq 0$ then $(a_1 - a_2)^2 = 0$
so we need $(k-1) > 0$.
Conclude $\langle \cdot | \cdot \rangle$ is an IP if $k > 1$.

PS6-5

5.3.1 Show $(A^*)^* = A$:

Defining property of A^*. For all $u, v \in V$:

$\langle u, Av \rangle = \langle A^* u, v \rangle$

So now compute

$$\langle u, A^* v \rangle = \langle (A^*)^* u, v \rangle$$

but $\langle u, A^* v \rangle = \overline{\langle A^* v, u \rangle} = \overline{\langle v, Au \rangle}$

by the defining property,

Since $\overline{\langle v, Au \rangle} = \langle Au, v \rangle$

we get:

$$\langle (A^*)^* u, v \rangle = \langle Au, v \rangle$$

for all u, v. This shows $(A^*)^* = A$.

This calculation holds for any IP space and does not use the matrix rep.

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PS6-C

5.3.3. $\varphi_1(x) = x_1 \quad \varphi_2(x) = x_2 \quad \varphi_3(x) = x_3$

$\langle x_\mu | x_\nu \rangle = \delta_{\mu\nu} \quad \{\varphi_i\}$ ONB

$A_m: \quad [A_m]_{ij} = \langle \varphi_i \, A_m \varphi_j \rangle$

$[A_1]_{ij} = \langle \varphi_i \, A_1 \varphi_j \rangle = \langle \varphi_i , \sum x_\ell \partial_\ell \varphi_j \rangle$

$\partial_\ell \varphi_j = \delta_{\ell j} \qquad [A_1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \langle \varphi_i , \varphi_j \rangle = \delta_{ij}$

$[A_2]_{ij} = \langle \varphi_i , (x_1 \partial_2 - x_2 \partial_1) \varphi_j \rangle$
$= \langle \varphi_i , x_1 \delta_{2j} - x_2 \delta_{1j} \rangle$
$= \delta_{i1}\delta_{2j} - \delta_{i2}\delta_{1j}$

$[A_2] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $\psi = x_1 - 2x_2 + 3x_3$ so $\psi = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

Compute $X = (A_1 - A_2)\psi = \sum_{i=1}^{3} x_i \partial_{x_i} \psi - (x_1 \partial_2 - x_2 \partial_1)\psi$

$= (x_1 - 2x_2 + 3x_3) + 2x_1 + x_2 = 3x_1 - x_2 + 3x_3$

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$$[A_1]^{-1}[A_2] = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[A_1]^{-1}[A_2] \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$ which is X.