PHY/MA 506

(p57.)

Solutions to Problem Set 7

5.6.1 $\varphi_1 = \alpha \quad \varphi_2 = \beta$ with $(\alpha,\alpha) = 1 = (\beta,\beta)$

$\mathcal{B} = \{\tilde{\alpha},\tilde{\beta}\}$ ONB of $\mathbb{C}^2$ $\quad (\alpha,\beta) = 0$

and write $\tilde{\alpha} = \begin{pmatrix}1\\0\end{pmatrix}$ $\tilde{\beta} = \begin{pmatrix}0\\1\end{pmatrix}$ in $\mathbb{C}^2$

a) $S_x\alpha = \frac{1}{2}\beta$
$S_x\beta = \frac{1}{2}\alpha$ $\implies$ matrix rep. of S_x in $\mathcal{B}$ basis

$$[S_x] = \begin{pmatrix} (\alpha, S_x\alpha) & (\alpha, S_x\beta) \\ (\beta, S_x\alpha) & (\beta, S_x\beta) \end{pmatrix}$$

The order of the elements is arbitrary $= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

$[S_x]\tilde{\alpha} = \tilde{\beta}$ etc.

$S_y\alpha = \frac{i}{2}\beta$
$S_y\beta = -\frac{i}{2}\alpha$ $\quad [S_y] = \begin{pmatrix} 0 & -i/2 \\ +i/2 & 0 \end{pmatrix}$

$S_z\alpha = \frac{1}{2}\alpha$
$S_z\beta = -\frac{1}{2}\beta$ $\quad [S_z] = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

The 3 2×2 matrices are the spin-$\frac{1}{2}$ matrices.
(these generate the Lie Algebra of $SU(2)$)

b) Change of basis: $\varphi_1' = c(\alpha+\beta)$ $\quad \varphi_2' = c(\alpha-\beta)$
Make $\{\varphi_1', \varphi_2'\}$ an ONB

$\|\varphi_1'\|^2 = 2c^2 = \|\varphi_2'\|^2$ so $c = \frac{1}{\sqrt{2}}$.

and $(\varphi_1', \varphi_2') = |c|^2(\alpha+\beta, \alpha-\beta)$
$= |c|^2(1-1) = 0$ so orthogonal.

P37.2

$$\varphi_1' = \frac{1}{\sqrt{2}}(\varphi_1 + \varphi_2) \qquad \varphi_2' = \frac{1}{\sqrt{2}}(\varphi_1 - \varphi_2)$$

c) Unitary map $\{\varphi_1, \varphi_2\} \xrightarrow{S} \{\varphi_1', \varphi_2'\}$

$$\varphi_i' = \sum_{j=1}^{2} S_{ij}\varphi_j \qquad S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$S^* = S^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = S^{-1}$$

Compute $\underset{m=x,y,z}{[S_m]^{B'}} = S\,[S_m]^{B}\,S^{-1}$

$$S_x' = S_z$$
$$S_y' = -S_y$$
$$S_z' = S_x$$

PS7-3

6.2.8. $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ $A^{*} = A$ all ev's real

eigenvalues

$$p_A(\lambda) = \det(A - \lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left((1-\lambda)^2 - 1\right) = (2-\lambda)(\lambda^2 - 2\lambda)$$

$$= -\lambda(\lambda-2)^2$$

$\sigma(A) = \{0, 2\}$ multiplicity of 0 is 1
multiplicity of 2 is 2

eigenvectors

$\lambda = 0$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a \\ b+c \\ b+c \end{pmatrix} = 0$ $a = 0$ $b = -c$

$V_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ normalized (easy to check that $A v_0 = 0$)

$\lambda = 2$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -b+c \\ b-c \end{pmatrix}$ a undetermined
$b = c$

General: $\begin{pmatrix} a \\ b \\ b \end{pmatrix}$ $a = 0$, $b = 1$ $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $A\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

and $a = 1$, $b = 0$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ a \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

PS7-4

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} \qquad v_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

<u>note</u> $(v_i, v_j) = \delta_{ij}$ so $\{v_1, v_2, v_3\}$ is an ONB of eigenvectors.

<u>Comment</u> Diagonalize A:
$$\begin{pmatrix}0 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix} = S\begin{pmatrix}2 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{pmatrix}S^{-1}$$

where
$$S = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\1 & 0 & 0\end{pmatrix} \qquad S^{-1} = \begin{pmatrix}0 & 0 & 1\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix}$$

Check: $SS^{-1} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ ✓

Then
$$SA = \begin{pmatrix}0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\0 & \sqrt{2} & \sqrt{2}\\2 & 0 & 0\end{pmatrix}$$

and
$$SAS^{-1} = \begin{pmatrix}0 & 0 & 0\\0 & \sqrt{2} & \sqrt{2}\\2 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}$$ ✓

P57-5

6.4.1 Similarity transf G invertible

$$A \to G A G^{-1}$$

$$\det(GAG^{-1} - \lambda I) = \det(GAG^{-1} - \lambda G G^{-1})$$

$$= \det G (A - \lambda I) G^{-1}$$

$$= \det G \cdot \det(A - \lambda I) \cdot \det G^{-1}$$

$$= (\det G)(\det G)^{-1} \cdot \det(A - \lambda I)$$

so the characteristic polynomials satisfy:

$$p_A(\lambda) = p_{GAG^{-1}}(\lambda)$$

and have the same zeros $\Rightarrow$ ev's of A = ev's of GAG^{-1}

<u>Comment</u>

$$\mathrm{Tr}\, GAG^{-1} = \mathrm{Tr}\, G^{-1} G A = \mathrm{Tr}\, A$$

by cyclicity of the trace: $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$

and $\det GAG^{-1} = \det A$

so both the trace & det of A are <u>invariant</u> under similarity transformations.

PS7-6

6.4.7 A, B self-adj and they have the same ev's:

There is a unitary S so $A_{diag} = SAS^{-1}$ where

$$A_{diag} = \begin{pmatrix} \lambda_1(A) & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2(A) & 0 & \cdots & \\ & & \ddots & & 0 \\ & 0 & & & \lambda_N(A) \end{pmatrix}$$

and there is a unitary T so

$$\underbrace{B_{diag} = TBT^{-1} = A_{diag}}$$

Then: $B = T^{-1} B_{diag} T = T^{-1} A_{diag} T$

$$= (T^{-1} S^{-1}) A (ST)$$

and $(ST)^{-1} = T^{-1} S^{-1} = T^* S^* = (ST)^*$

so $U = ST$ is unitary and

$$B = U^* A U$$

showing that A & B are unitarily equivalent.

(137-49)

6.5.3. U unitary

If $Uv = \lambda v$ then $(Uv, Uv) = |\lambda|^2 \|v\|^2$

But $(Uv, Uv) = (v, U^*Uv) = \|v\|^2$

So $\|v\|^2 = |\lambda|^2 \|v\|^2 \Rightarrow |\lambda| = \underline{1}$

The eigenvalues of a unitary matrix have modulus $\underline{1}$:

$\lambda_j = e^{i\phi_j}$ for $\phi_j \in \mathbb{R}$.