1. Consider the Legendre operator \( L \) on \( L^2([-1,1]) \) defined by
\[
Ly = -(1-x^2)y'' + 2xy'.
\]
Show that \( L \) is self-adjoint by writing \( L \) in self-adjoint form and show that the boundary terms vanish with no other conditions on the functions. There are solutions to \( Lu = \lambda u \) that are finite at \( x = \pm 1 \). These only occur if \( \lambda = \ell(\ell+1) \), where \( \ell = 0, 1, 2, \ldots \). Why are these Legendre polynomials an orthonormal basis for \( L^2([-1,1]) \)?

2. Show that the linear operator \( L = -d^2/dx^2 \) on \( L^2([0,2\pi]) \) is hermitian on the functions that satisfy periodic boundary conditions: \( y(0) = y(2\pi) \) and \( y'(0) = y'(2\pi) \), and that are twice differentiable. That is, for any two such functions
\[
\int_0^{2\pi} f(x)(Lg)(x) \, dx = \int_0^{2\pi} (Lf)(x) g(x) \, dx.
\]
Find the normalized eigenfunctions of \( L \), that is, functions satisfying \( Lf = \lambda f \), with these properties, and the corresponding eigenfunctions. Check that the eigenfunctions are orthogonal.

3. Consider the nonhomogeneous BVP: \( y'' = x(x-2\pi) \) on \([0,\pi]\). Expand \( y \) in the eigenfunctions of the related Sturm-Liouville problem \( Ly = -y'' = \lambda y \) with DBC at 0 and \( \pi \). Expand \( h(x) = x(x-2\pi) \) in the eigenfunctions of this Sturm-Liouville problem. Find a formal series solution for \( y \).

4. Find the Fourier series for a square wave:
\[
f(x) = \begin{cases} 
  h/2 & 0 < x < \pi \\
  -h/2 & -\pi < x < 0 
\end{cases}
\]
What is the value of the series at \( x = -\pi, 0, \pi \)? Is this reasonable?