MA/PHY506 Fall 2018
Problem Set 8
DUE: 19 November 2018

1. Consider the Legendre operator L on $L^2([-1,1])$ defined by

$$Ly = -(1-x^2)y'' + 2xy'.$$

Show that L is self-adjoint by writing L in self-adjoint form and show that the boundary terms vanish with no other conditions on the functions. There are solutions to $Lu = \lambda u$ that are finite at $x = \pm 1$. These only occur if $\lambda = \ell(\ell+1)$, where $\ell = 0, 1, 2, \ldots$. Why are these Legendre polynomials an orthonormal basis for $L^2([-1,1])$?

2. Show that the linear operator $L = -d^2/dx^2$ on $L^2([0, 2\pi])$ is hermitian on the functions that satisfy periodic boundary conditions: $y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$, and that are twice differentiable. That is, for any two such functions

$$\int_0^{2\pi} \overline{f}(x)(Lg)(x)\, dx = \int_0^{2\pi} \overline{Lf}(x) g(x)\, dx.$$

Find the normalized eigenfunctions of L, that is, functions satisfying $Lf = \lambda f$, with these properties, and the corresponding eigenfunctions. Check that the eigenfunctions are orthogonal.

3. Consider the nonhomogeneous BVP: $y'' = x(x-2\pi)$ on $[0, \pi]$. Expand y in the eigenfunctions of the related Sturm-Liouville problem $Ly = -y'' = \lambda y$ with DBC at 0 and π. Expand $h(x) = x(x-2\pi)$ in the eigenfunctions of this Sturm-Liouville problem. Find a formal series solution for y.

4. Find the Fourier series for a square wave:

$$f(x) = \begin{cases} h/2 & 0 < x < \pi \\ -h/2 & -\pi < x < 0 \end{cases}$$

What is the value of the series at $x = -\pi, 0, \pi$? Is this reasonable?