MAPHY 506

PS8-1

Solutions to PS 8

1. $Ly = -(1-x^2)y'' + 2xy'$ on $\mathcal{H} = L^2([-1,1])$

show $(f, Lg) = (Lf, g)$ for some fncs. $f, g \in \mathcal{H}$.

<u>first</u> writing $L = a_2 D_x^2 + a_1 D_x$

with $a_2 = -(1-x^2)$ & $a_1 = 2x$

$\frac{da_2}{dx} = 2x = a_1$

$p(x) = -(1-x^2)$

a condition so $L = -D_x(1-x^2)D_x$

as is easily checked.

We already verified for $L = D_x p D_x$

$(f, Lg) = (Lf, g) +$ Boundary Term.

Boundary Term:

$$-(1-x^2)\left(\bar{f}(x)g'(x) - \bar{f}'(x)g(x)\right)\Big|_{-1}^{1} = 0$$

Since this vanishes due to the vanishing of $p(\pm 1) = 0$,

$\sigma(L)$ is a set of ev's going to ∞. $Lf_\ell = \ell(\ell+1)f_\ell$

with polynomial solns. $f_\ell(x)$.

(P58-2)

2. $Lf = -f''$ on $L^2([0,2\pi])$ with PBC: $y(0) = y(2\pi)$
$Ly = \lambda^2 y$ L is self-adj on these $y'(0) = y'(2\pi)$
$y(x) = A\cos\lambda x + B\sin\lambda x$ fncs.
$y'(x) = -\lambda A\sin\lambda x + \lambda B\cos\lambda x$

Fix BC: $\begin{cases} y(0) = A = y(2\pi) = A\cos 2\pi\lambda + B\sin 2\pi\lambda \\ y'(0) = \lambda B = y'(2\pi) = -\lambda A\sin 2\pi\lambda + \lambda B\cos 2\pi\lambda \end{cases}$

so $A(1-\cos 2\pi\lambda) = B\sin 2\pi\lambda$
$\lambda B(1-\cos 2\pi\lambda) = -\lambda A\sin 2\pi\lambda$

Check various cases:

(*) $\begin{cases} A=0,\ B\neq 0: & \lambda(1-\cos 2\pi\lambda) = \sin 2\pi\lambda = 0 \\ & \text{If } \lambda\in\mathbb{N} \text{ we get } 0=0 \text{ so this is a solution} \\ A\neq 0,\ B=0: & \left.\begin{array}{l} 1-\cos 2\pi\lambda = 0 \\ \sin 2\pi\lambda = 0 \end{array}\right\} \text{again } \lambda = n \end{cases}$

For $\lambda = n \geq 1$, $\psi_n(x) = \cos nx$
$\varphi_n(x) = \sin n\pi$
are eigenfncs. and LI.

For $\lambda = n = 0$ $\psi_0(x) = 1$ is an ef.

$\sigma(L) = \{n \mid n = 0,1,2,\ldots\}$ with multiplicity 2, $n\geq 1$ and 1 for $n=0$.

with ef: $\psi_0(x) = 1$
$\psi_n(x) = \cos nx$
$\varphi_n(x) = \sin nx$.

(*) One should write:

$$\begin{pmatrix} 1-\cos 2\pi\lambda & -\sin 2\pi\lambda \\ \lambda\sin 2\pi\lambda & \lambda(1-\cos 2\pi\lambda) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Want λ so the det. is zero: $\lambda = 0$ is a soln. and

$\lambda(1-\cos 2\pi\lambda)^2 + \lambda\sin^2 2\pi\lambda = 0 = \lambda(1-2\cos 2\pi\lambda) + \lambda = 2\lambda(1-\cos 2\pi\lambda)$

so $\lambda = n$.

(P58-3)

3. Solve $y'' = x(x-2\pi)$ on $[0,\pi]$
with DBC.

Homog. problem: $Ly = \lambda^2 y \quad y(0) = y(\pi) = 0$

$Ly = -y''$

$$\begin{cases} \lambda^2 = n^2 \\ \psi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx \quad \text{ONB} \end{cases}$$

$$\int_0^\pi \psi_n(x)^2 dx = \int_0^\pi \sin^2 nx \, dx = \frac{\pi}{2}$$

Nonhomog. problem:

$Ly = x(2\pi - x)$ (change sign)

Expand $h(x) = x(2\pi - x) = \sum_{n=1}^\infty h_n \psi_n(x)$

ef expansion.

$$h_n = \sqrt{\frac{2}{\pi}} \int_0^\pi x(2\pi - x) \sin nx \, dx$$

Expand $y(x) = \sum_{n=1}^\infty c_n \psi_n(x)$

So $L\left(\sum_{n=1}^\infty c_n \psi_n\right) = \sum_{n=1}^\infty c_n n^2 \psi_n = \sum_{n=1}^\infty h_n \psi_n$

Take the IP with ψ_m:

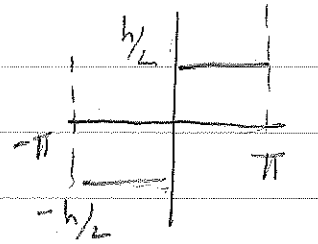
$n^2 c_n = h_n$ for all $n \geq 1$

$c_n = \frac{h_n}{n^2}$

So $y_p(x) = \sum_{n=1}^\infty \sqrt{\frac{2}{\pi}} \left(\frac{h_n}{n^2}\right) \sin(nx)$.

PS 8-4

4. Fourier series.



$$f(x) = \begin{cases} h/2 & 0 < x < \pi \\ -h/2 & 0 > x > -\pi \end{cases}$$

General FS: $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos kx \, f(x)\, dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin kx \, f(x)\, dx$$

f is odd so $a_k = 0$ $k = 0, 1, 2, \ldots$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{0} \frac{-h}{2} \sin kx \, dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{h}{2} \sin kx \, dx$$

$$= \frac{-h}{2\pi}\left(\frac{-\cos kx}{k}\Big|_{-\pi}^{0}\right) + \frac{h}{2\pi}\left(\frac{-\cos kx}{k}\Big|_{0}^{\pi}\right)$$

$$= \frac{-h}{2\pi k}\left[(-1)^k - 1\right] + \frac{h}{2\pi k}\left[1 - (-1)^k\right]$$

$$= \frac{h}{\pi k}\left[1 - (-1)^k\right] = \begin{cases} 0 & k \text{ even} \\ \frac{2h}{\pi k} & k \text{ odd} \end{cases}$$

$$\boxed{f(x) = \sum_{j=1}^{\infty} \frac{2h}{\pi(2j-1)} \sin(2j-1)x}$$

$f(0) = 0$

and

$f(x = \pm\pi) = 0 = \lim_{\varepsilon \to 0} \frac{1}{2}\left[f(\pi - \varepsilon) + f(-\pi + \varepsilon)\right]$ average value.