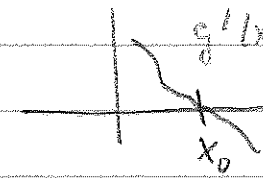
Solutions to PS 9 (PS9-1)

1. $$\int_{-\infty}^{\infty} f(x)\,\delta[g(x)]\,dx \qquad g(x_0)=0 \quad g'(x_0)\neq 0$$

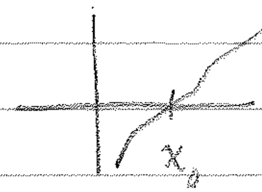


$u = g(x)$

$du = g'(x)\,dx$

g is one-to-one near x_0 so $u = g^{-1}(x)$
and $0 = g^{-1}(x_0)$

If $g'(x) > 0$ x near x_0 then:



$$\int_{x=-\infty}^{\infty} f(x)\,\delta[g(x)]\,dx = \int_{u=0-\varepsilon}^{0+\varepsilon} f(g^{-1}(u))\,\delta(u)\,\frac{du}{g'(g^{-1}(u))}$$

$$= \frac{f(g^{-1}(0))}{g'(g^{-1}(0))} = \frac{f(x_0)}{g'(x_0)}$$

If $g'(x) < 0$ for x near x_0 then:

$$\int_{x=-\infty}^{\infty} f(x)\,\delta[g(x)]\,dx = \int_{u=0+\varepsilon}^{0-\varepsilon} f(g^{-1}(u))\,\delta(u)\,\frac{du}{g'(g^{-1}(u))}$$

note change of endpts.

$$= \int_{0-\varepsilon}^{0+\varepsilon} f(g^{-1}(u))\,\delta(u)\,\frac{du}{|g'(g^{-1}(u))|}$$

$$= \frac{f(x_0)}{|g'(x_0)|}$$

(p59-2)

2. $$f(x) = \begin{cases} 0 & x<0 \\ Ae^{-\alpha x} & x>0 \end{cases} \quad \alpha>0. \quad A\neq 0$$

$$\hat{f}(k) = \frac{A}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} e^{-\alpha x} e^{-ikx}\,dx$$

$$= \frac{A}{(2\pi)^{\frac{1}{2}}} \int_0^{\infty} e^{-(\alpha+ik)x}\,dx$$

$$= \frac{A}{(2\pi)^{\frac{1}{2}}} \left[\frac{e^{-(\alpha+ik)x}}{-(\alpha+ik)} \right]_{x=0}^{\infty}$$

$$= \frac{A}{(2\pi)^{\frac{1}{2}}} \left(\frac{1}{\alpha+ik} \right) \quad \text{since } e^{-\alpha x} \to 0 \text{ as } x \to \infty$$

$$\hat{f}(k) = \frac{A}{(2\pi)^{\frac{1}{2}}} \left[\frac{(\alpha-ik)}{\alpha^2+k^2} \right]$$

PS9-3

3. $$\int_{-\infty}^{\infty} \delta'(x-x_0) f(x)\,dx = \lim_{M\to\infty} \underbrace{f(x)\,\delta(x-x_0)\Big|_{-M}^{M}} - \int_{-\infty}^{\infty} \delta(x-x_0) f'(x)$$

Integrate by parts $\qquad$ Vanishes as $\delta(M-x_0)=0$ for M large!

$$dv = \delta'(x-x_0)\,dx \Rightarrow v = \delta(x-x_0)$$
$$u = f(x) \qquad du = f'(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta'(x-x_0) f(x)\,dx = -\int_{-\infty}^{\infty} \delta(x-x_0) f'(x)\,dx$$
$$= -f'(x_0)$$

4. $$\hat{\psi}(k) = \frac{C_0}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} e^{-ik\cdot x} e^{-\alpha\|x\|}\, d^3x$$

Use (r,θ,ϕ) with $\theta \in [-\pi,\pi]$, $\phi \in (0,2\pi)$

$\|x\| = r$ and $k\cdot x = r\|k\|\cos\theta$

$d^3x = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$\hat{\psi}(k) = \frac{C_0}{(2\pi)^{3/2}} \int_0^{\infty} r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi\; e^{-\alpha r} e^{-i\|k\| r\cos\theta}$$

$$\int_0^{\pi} e^{-i\|k\|r\cos\theta} \sin\theta\, d\theta = \frac{e^u}{i\|k\|r}\Big|_{-i\|k\|r}^{i\|k\|r}$$

$u = -i\|k\|r\cos\theta$

$du = +i\|k\|r\sin\theta\, d\theta$

$$= \frac{e^{i\|k\|r} - e^{-i\|k\|r}}{i\|k\|r} = \frac{2\sin(\|k\|r)}{\|k\|r}$$

PSE 4

$$\hat{\psi}(k) = \frac{2C_0 (2\pi)^{-\frac{1}{2}}}{\|k\|} \int_0^\infty r \sin(\|k\| r)\, e^{-\alpha r}\, dr$$

$$= \frac{2C_0 (2\pi)^{-\frac{1}{2}}}{\|k\|} \left\{ \frac{1}{2i} \int_0^\infty \left[e^{(i\|k\|-\alpha) r} - e^{(-i\|k\|-\alpha) r} \right] r\, dr \right\}$$

$$\int_0^\infty e^{-\beta r} r\, dr = -\frac{d}{d\beta} \int_0^\infty e^{-\beta r}\, dr = -\frac{d}{d\beta}\left(\frac{e^{-\beta r}}{-\beta}\right)\Big|_0^\infty$$

$$= -\frac{d}{d\beta}\left(\frac{1}{\beta}\right) \quad \frac{1}{\beta^2}$$

$$\beta = \alpha \mp i\|k\|$$

$$= \frac{-i C_0 (2\pi)^{-\frac{1}{2}}}{\|k\|} \left[\left(\frac{1}{\alpha - i\|k\|}\right)^2 - \left(\frac{1}{\alpha + i\|k\|}\right)^2 \right] = \frac{-i C_0 (2\pi)^{-\frac{1}{2}}}{\|k\|} \left[\frac{(\alpha + i\|k\|)^2 - (\alpha - i\|k\|)^2}{(\alpha^2 + \|k\|^2)^2} \right]$$

$$= \frac{-i C_0 (2\pi)^{-\frac{1}{2}}}{\|k\|} \left[\frac{4 i \|k\| \alpha}{(\alpha^2 + \|k\|^2)^2} \right] = \frac{4 C_0 \alpha}{(2\pi)^{\frac{1}{2}} (\alpha^2 + \|k\|^2)^2}$$

Note that it is real.