MA/PHY 506 Fall 2018
Midterm Exam - 100 Points
26 October 2018

INSTRUCTIONS: PLEASE WORK ALL FOUR PROBLEMS BELOW. NO BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	30	
2	20	
3	30	
4	20	
TOTAL	100	

Problem 1. (30 points.) A form of the hypergeometric ODE is:

$$x(x-1)y''(x) + 3xy'(x) + y(x) = 0. \tag{1}$$

i. Write the ODE in standard form. Classify the finite singular points, if any.

ii. Apply the Frobenius method to this ODE around $x_0 = 0$. Derive the indicial equation and the recursion relation. You do not have to solve the recursion relation.

iii. Find the Wronskian of two independent solutions to this hypergeometric ODE.

$$y''(x) + \frac{3}{x-1} y'(x) + \frac{1}{x(x-1)} y(x) = 0$$

$x_0 = 0$ $\quad \lim_{x\to 0} xP(x) = 0, \quad \lim_{x\to 0} x^2 Q(x) = 0 \qquad x=0$ reg. sing. pt,

$x_0 = 1$ $\quad \lim_{x\to 1} (x-1)P(x) = 3 \quad \lim_{x\to 1} (x-1)^2 Q(x) = 0 \qquad$ reg. sng. pt.

$$y(x) = \sum_{j=0}^{\infty} a_j x^{j+s} \qquad y'(x) = \sum (j+s)x^{j+s-1} a_j \qquad y''(x) = \sum (j+s)(j+s-1) x^{j+s-2} a_j$$

$$\sum_j \{ (j+s)(j+s-1) x^{j+s} a_j - (j+s)(j+s-1) x^{j+s-1} a_j$$
$$+ 3(j+s) x^{j+s} a_j + x^{j+s} a_j \} = 0.$$

Re-index $\displaystyle\sum_{j=0}^{\infty} (j+s)(j+s-1)x^{j+s-1} a_j = \sum_{m=0}^{\infty} (m+s)(m+1+s) x^{m+s} a_{m+1}$

$m = j-1$
$j = m+1$

$$+ s(s-1)x^{s-1} a_0$$

$$\sum_{j=0}^{\infty} \{ [(j+s)(j+s-1) + 3(j+s) + 1] a_j - (j+s)(j+s+1) a_{j+1} \} x^{j+s} +$$

$$+ s(s-1) x^{s-1} a_0 = 0$$

$$\begin{cases} \text{Indicial eqn: } & s(s-1) = 0 \quad (a_0 \neq 0) \quad \text{roots } s=1 \text{ or } s=0. \\ \text{Recursion: } & a_{j+1} = \dfrac{(j+s)(j+s+2)+1}{(j+s)(j+s+1)} a_j \end{cases}$$

Problem 2. (20 points.) Suppose that a linear transformation T acts on the orthonormal basis $\mathcal{B} = \{v_1, v_2, v_3\}$ of a real linear vector space V by:

$$\begin{aligned} Tv_1 &= v_1 - v_2 + v_3 \\ Tv_2 &= -v_1 + v_3 \\ Tv_3 &= -v_2 + v_3 \end{aligned}$$

i. Write the matrix representation $[T]^{\mathcal{B}}$ of T relative to the ON basis $\mathcal{B}$.

ii. What type of matrix is $[T]^{\mathcal{B}}$?

iii. Is the set of vectors $\{Tv_1, Tv_2, Tv_3\}$ linearly independent?

iv. Is the set of vectors $\{Tv_1, Tv_2, Tv_3\}$ an orthonormal basis of V? If not, construct an orthornomal set of vectors that is a basis for the span of $\{Tv_1, Tv_2, Tv_3\}$.

i. $(T)^{\mathcal{B}}_{ij} = (v_i, Tv_j)$ so $[T]^{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

ii. $\det T = (-1)(-1)\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} + (-1)(-1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -1 + 2 = 1 \neq 0$. 2^{nd} row expansion

so T is <u>invertible</u>.

iii. The set $\{Tv_1, Tv_2, Tv_3\}$ is LI since T is invertible. It spans V since

iv $\dim V = 3$. Since $[T]^{\mathcal{B}}$ isn't orthogonal, the basis isn't ON.

Use Gram-Schmidt: $w_1 = \dfrac{Tv_1}{\|v_1\|} = \dfrac{\sqrt{3}}{3}(v_1 - v_2 + v_3)$

w_2: let $u_2 = Tv_2 - (w_1, Tv_2)w_1$ so $u_2 \perp w_1$.

But $(w_1, Tv_2) = \dfrac{\sqrt{3}}{3}(-1+1) = 0$ so $u_2 = Tv_2$

$\|u_2\| = \sqrt{2}$

$\Rightarrow w_2 = \dfrac{\sqrt{2}}{2}(-v_1 + v_3)$.

For w_3, let $u_3 = Tv_3 - (w_1, Tv_3)w_1 - (w_2, Tv_3)w_2$. $(w_1, Tv_3) = \dfrac{2\sqrt{3}}{3}$

$(w_2, Tv_3) = \dfrac{\sqrt{2}}{2}$

$= (-v_2 + v_3) - \dfrac{\sqrt{3}}{3}(v_1 - v_2 + v_3) \cdot \dfrac{2\sqrt{3}}{3} - \dfrac{1}{2}(-v_1 + v_3)$

$= -\dfrac{1}{6}v_1 - \dfrac{1}{3}v_2 - \dfrac{1}{6}v_3$

$\|u_3\|^2 = \dfrac{1}{6}$

$$\left.\begin{aligned} w_3 &= -\frac{\sqrt{6}}{6}(v_1 + 2v_2 + v_3) \\ w_2 &= \frac{\sqrt{2}}{2}(-v_1 + v_3) \\ w_1 &= \frac{\sqrt{3}}{2}(v_1 - v_2 + v_3) \end{aligned}\right\}$$

$\{w_1, w_2, w_3\}$ ONB of V

Continue work on Problem 1.

$$W(x) = Ce^{-\int^{x} P(s)ds}$$

$$P(s) = \frac{3}{s-1}$$

$$\int^{x} \frac{3ds}{s-1} = 3\log(x-1)$$

$$e^{-\int^{x} P(s)ds} = e^{\log(x-1)^{-3}}$$

so $\boxed{W(x) = \frac{c}{(x-1)^3}}$

Problem 3. (30 points.) Find the most general solution to the ODE

$$t^2y''(t) - t(t+2)y'(t) + (t+2)y(t) = t^3. \quad t > 0.$$

i. Find a solution of the homogeneous ODE of the form $y(t) = t^m$ and then construct a second independent solution.

ii. Find the Green's function associated with the ODE.

iii. Find a particular solution and write the most general solution of the ODE. Make sure to put the ODE in standard form.

i. $t^2y'' - t(t+2)y' + (t+2)y =$ $\quad t > 0$

$y'' - \frac{(t+2)}{t}y' + \frac{(t+2)}{t^2}y = t$

$y'(t) = mt^{m-1}$

$y''(t) = m(m-1)t^{m-2}$

$m(m-1)t^m - m(t+2)t^m + (t+2)t^m = 0$

$m(m-1) - mt - 2m + t + 2 = 0$

$t(1-m) = 0 \quad m = 1$

Check $\quad -t - 2 + t + 2 = 0$

$y_1(t) = t$

$y_2(t) = y_1(t)\int^t \frac{e^{-\int^s p(u)\,du}}{y_1(s)^2}$

$p(u) = -\frac{u(u+2)}{u^2} = -1 - \frac{2}{u}$

$\int^s p(u)\,du = -s - 2\log s$

$e^{-\int^s p(u)du} = e^s e^{\log s^2} = s^2e^s.$

$W(t) = \begin{vmatrix} t & te^t \\ 1 & (1+t)e^t \end{vmatrix}$

$= t^2e^t$

$y_2(t) = t\int^t \frac{s^2e^s}{s^2} = te^t$

$\boxed{\text{Basis of soln sp. } \{y_1(t) = t,\ y_2(t) = te^t\}}$

$y_1(t) = t \quad y_2(t) = te^t \quad h(s) = t^2 \quad W(t) = t^2 e^t$

Continue work on Problem 3.

$$G_2(t,s) = \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{W(s)} = \frac{ts(e^t - e^s)}{s^2 e^s}$$

$$\boxed{G_2(t,s) = \frac{t}{s}\left[e^{t-s} - 1\right]}$$

$$y_p(t) = \int^t G_2(t,s)h(s)\,ds = t\int^t \frac{1}{s}(e^{t-s})\, s\,ds - t\int_0^t \frac{1}{s} s^2\,ds$$

$$= te^t \int^t e^{-s}\,ds - t\int^t ds$$

$$= te^t(-e^{-t}) - t^2$$

$$= -t - t^2$$

solⁿ of homog.

$$\boxed{y_p(t) = -t^2}$$

$$y_p'(t) = -2t$$

$$y_p''(t) = -2$$

$$t^2 y_p'' - t^2 y_p' - 2t y_p' + t y_p + 2y_p = -2t^2 + 2t^3 + 4t^2 - t^3 - 2t^2$$

$$= t^3$$

General solⁿ:

$$\boxed{y_g(t) = At + Bte^t - t^2}$$

Problem 4. (20 points.) Find the solution to the initial value problem:

$$y'(t) = 2t(1 + y(t)),$$

and $y(t = 0) = 0$.

$$y'(t) - 2ty(t) = 2t$$

Integrating factor:

$$\mu(t) = e^{\int^t p(s)ds} = e^{-t^2}$$

$$p(s) = -2s$$

$$\frac{d}{dt}(\mu y) = \mu g = 2te^{-t^2}$$

$$\mu(t)y(t) = \int^t 2se^{-s^2}ds = -e^{-t^2} + C$$

General Soln. $\boxed{y(t) = -1 + Ce^{t^2}}$

Unique solution: $y(t=0) = 0$

$$y(0) = -1 + C = 0, \quad C = 1$$

$$\boxed{y(t) = -1 + e^{t^2}, y(0) = 0}$$

Check: $y'(t) = 2te^{t^2} = 2t(y+1)$ ✓

Can also separate

$$\frac{dy}{1+y} = 2t\,dt$$