PHY/MA 507

S2013

PS2-1

Solutions to PS 2

11.2.1 Is $f(z) = x$ analytic?

Check C-R eqns. $u(x,y) = x \quad v(x,y) = 0$

$$\frac{\partial u}{\partial x} = 1 \stackrel{?}{=} \frac{\partial v}{\partial y} = 0$$ No - so it can't be analytic. Another way to think of this:

$f(z) = \frac{1}{2}(z + \bar{z})$ so although $\frac{1}{2}z$ is analytic $g(z) = \bar{z}$ is not (again CR eqns. don't hold)

11.2.3 Find the harmonic conjugate: $f(z) = u(x,y) + i v(x,y)$ then u, v satisfy CR $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$:

a) $u(x,y) = x^3 - 3xy^2$ ① $\partial_x u = 3x^2 - 3y^2 = \partial_y v$

Integrate: $v(x,y) = 3x^2 y - y^3 + h(x)$

② $\partial_y u = -6xy = -\partial_x v \Rightarrow \quad v(x,y) = 3x^2 y + g(y)$

Compare $v(x,y) = 3x^2 y - y^3$. Double check:

$\partial_x u = 3x^2 - 3y^2 \stackrel{?}{=} \partial_y v = 3x^2 - 3y^2$ ✓

$\partial_y u = -6yx \stackrel{?}{=} -\partial_x v = -6xy$ ✓

$$\boxed{v(x,y) = 3x^2 y - y^3}$$

b) $v(x,y) = e^{-y}\sin x$ ① $\partial_x v = e^{-y}\cos x = -\partial_y u$

so $u(x,y) = e^{-y}\cos x + h(x)$

② $\partial_y v = -e^{-y}\sin x = \partial_x u \Rightarrow u(x,y) = e^{-y}\cos x + g(y)$

Compare: $\boxed{u(x,y) = e^{-y}\cos x}$

Note $f(z) = e^{-y}(\cos x + i \sin x) = e^{ix-y} = e^{i(x+iy)}$
$= e^{z}$.

11.2.6. $f(z) = R(r,\theta)e^{i\Theta(r,\theta)}$

$z \to \delta z + z$ w/ $z = re^{i\theta}$ Compute δz 2 ways:

① $z + \delta z = (r + \delta r)e^{i\theta} = z + \delta r e^{i\theta}$ so $\delta z = \delta r e^{i\theta}$

$$\frac{d}{dz}f(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$ psz-2

$$f(r+\delta r, \theta) - f(r,\theta) = R(r+\delta r, \theta) e^{i\Theta(r+\delta r,\theta)} - R(r,\theta) e^{i\Theta(r,\theta)}$$

$$= [R(r+\delta r, \theta) - R(r,\theta)] e^{i\Theta(r+\delta r,\theta)} + R(r,\theta)\left[e^{i\Theta(r+\delta r,\theta)} - e^{i\Theta(r,\theta)}\right]$$

take $\delta r \to 0$

$$\frac{df}{dz} = \frac{\partial R}{\partial r} e^{i\Theta} + R(r,\theta) e^{i\Theta}\, i \frac{\partial \Theta}{\partial r}$$

$$= \left(\frac{\partial R}{\partial r} + iR\frac{\partial \Theta}{\partial r}\right) e^{i\Theta} e^{-i\theta}$$

(2) $\delta z = (ir\,\delta\theta) e^{i\theta}$

$$f(r, \theta+\delta\theta) - f(r,\theta) = R(r,\theta+\delta\theta) e^{i\Theta(r,\theta+\delta\theta)} - R(r,\theta) e^{i\Theta(r,\theta)}$$

$$\lim_{\delta\theta\to 0}\left(\frac{f(r,\theta+\delta\theta) - f(r,\theta)}{ir\,\delta\theta}\right) e^{-i\theta}$$

$$= \lim_{\delta\theta\to 0}\left\{\frac{-i}{r}\left(\frac{R(r,\theta+\delta\theta) - R(r,\theta)}{\delta\theta}\right) e^{i\Theta(r,\theta,+\delta\theta)} e^{-i\theta} + e^{-i\theta}\frac{R(r,\theta)}{ir}\left(\frac{e^{i\Theta(r,\theta+\delta\theta)} - e^{i\Theta(r,\theta)}}{\delta\theta}\right)\right\}$$

$$\frac{df}{dz} = \left[\frac{-i}{r}\frac{\partial R}{\partial \theta} + \frac{R}{r}\frac{\partial \Theta}{\partial \theta}\right] e^{i\Theta} e^{-i\theta}$$

Compare (1) and (2) :

$$\boxed{\begin{aligned} &\bullet\ \frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta} \\ &\bullet\ -\frac{1}{r}\frac{\partial R}{\partial \theta} = R\frac{\partial \Theta}{\partial r} \end{aligned}}$$

11.2.7. Differentiate the second CR:

Left side

(1) $$\frac{\partial}{\partial r}\left(R\frac{\partial \Theta}{\partial r}\right) = \frac{\partial R}{\partial r}\frac{\partial \Theta}{\partial r} + R\frac{\partial^2 \Theta}{\partial r^2} = \frac{1}{r^2}\frac{\partial R}{\partial \theta} - \frac{1}{r}\frac{\partial^2 R}{\partial \theta \partial r}$$

For the mixed term differentiate the first CR eqn:

(2) $$\frac{\partial^2 R}{\partial r \partial \theta} = \frac{1}{r}\frac{\partial R}{\partial \theta}\frac{\partial \Theta}{\partial \theta} + \frac{R}{r}\frac{\partial^2 \Theta}{\partial \theta^2}$$

PS2-3

Substitute this into (1):

$$R\frac{\partial^2 \Theta}{\partial r^2} = \frac{1}{r^2}\frac{\partial R}{\partial \theta} - \frac{1}{r}\left[\frac{1}{r}\frac{\partial R}{\partial \theta}\frac{\partial \Theta}{\partial \theta} + \frac{R}{r}\frac{\partial^2 \Theta}{\partial \theta^2}\right] - \frac{\partial R}{\partial r}\frac{\partial \Theta}{\partial r}$$

Reorganize:

$$R\left[\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2}\right] = \frac{1}{r^2}\left(-rR\frac{\partial \Theta}{\partial r}\right) - \frac{1}{r^2}\frac{\partial \Theta}{\partial \theta}\left(-Rr\frac{\partial \Theta}{\partial r}\right.$$

$$-\frac{\partial \Theta}{\partial r}\left(\frac{R}{r}\frac{\partial \Theta}{\partial \theta}\right)$$ (use CR to eliminate all partials in R)

$$\Rightarrow \quad \frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} + \frac{1}{r}\frac{\partial \Theta}{\partial r} = 0$$

11.2.9 a) $f(z) = \frac{\sin z}{z}$ This is actually continuous at $z=0$ if we define $f(0)=1$

$\frac{df}{dz} = \frac{\cos z}{z} - \frac{\sin z}{z^2}$ exists on $\mathbb{C}\setminus\{0\}$ but can be extended to $\mathbb{C}$.

so f analytic on $\mathbb{C}\setminus\{0\}$

b) $f(z) = \frac{1}{z^2-1}$ $\quad \frac{df}{dz} = \frac{-2z}{(z^2-1)^2} = \frac{-2z}{(z-1)^2(z+1)^2}$

singular at $z=1$ & $z=-1$ so f is analytic on $\mathbb{C}\setminus\{-1,+1\}$.

c) $f(z) = \tan z = \frac{\sin z}{\cos z}$ $\quad f'(z) = \frac{\cos z}{\cos z} + \frac{\sin z \cdot \sin z}{(\cos z)^2} = \sec^2 z$

f is defined except where $\cos z = 0$.

$\cos z = \cos x \cosh y - i \sin x \sinh y = 0$ $\quad \cos x \cosh y = 0$

$\sin x \sinh y = 0 \Rightarrow y = 0$ $\quad \Rightarrow x = \left(\frac{2k+1}{2}\right)\pi$

f analytic on $\mathbb{C}\setminus\left\{\frac{(2k+1)\pi}{2} \mid k \in \mathbb{Z}\right\}$

52.5

Now use (*) for the L & R sides

$$\frac{\partial u}{\partial r} = \frac{\partial R}{\partial r}\cos\Theta - R\sin\Theta\,\frac{\partial \Theta}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$= \frac{1}{r}\left(\frac{\partial R}{\partial \theta}\sin\Theta + R\cos\Theta\,\frac{\partial \Theta}{\partial \theta}\right)$$

$$\Rightarrow \left(\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial \Theta}{\partial \theta}\right)\cos\Theta = \left(\frac{1}{r}\frac{\partial R}{\partial \theta} + R\frac{\partial \Theta}{\partial r}\right)\sin\Theta$$

Conclude the coefficients must vanish:

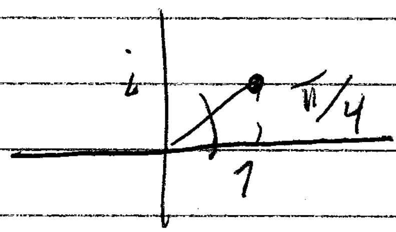
$$\boxed{\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta} \quad \text{and} \quad R\frac{\partial \Theta}{\partial r} = -\frac{1}{r}\frac{\partial R}{\partial \theta}}$$

PS2-6

2 a) $e^{2+i} = e^2(\cos 1 + i \sin 1)$

$= e^2 \cos 1 + i\, e^2 \sin 1$

b) $(1+i)^{1+i} = e^{(1+i)\operatorname{Log}_{PB}(1+i)}$



$\arg_{PB}(1+i) = \pi/4$

$|1+i| = \sqrt{2}$

$$\operatorname{Log}_{PB}(1+i) = \log|1+i| + i \arg_{PB}(1+i)$$

$$= \log\sqrt{2} + i\pi/4$$

$$(1+i)\operatorname{Log}_{PB}(1+i) = (1+i)(\log\sqrt{2} + i\pi/4)$$

$$= (\log\sqrt{2} - \pi/4) + i(\log\sqrt{2} + \pi/4)$$

$$e^{(1+i)\operatorname{Log}_{PB}(1+i)} = e^{(\log\sqrt{2} - \pi/4)} e^{i(\log\sqrt{2} + \pi/4)}$$

$$= \sqrt{2}\, e^{-\pi/4}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\left(\cos\log\sqrt{2} + i \sin\log\sqrt{2}\right)$$

$$= \sqrt{2}\, e^{-\pi/4}\left\{\frac{\sqrt{2}}{2}\left[\cos(\log\sqrt{2}) - \sin(\log\sqrt{2})\right] + i\frac{\sqrt{2}}{2}\left[\cos(\log\sqrt{2}) + \sin(\log\sqrt{2})\right]\right\}$$

WOW!

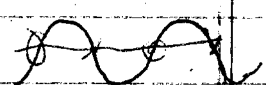
ps2-7

3. $\cos z = \frac{1}{2} = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)$

$\Rightarrow 1 = (cx + isx)\,e^{-y} + (cx - isx)\,e^{y}$

$= 2\cos x \cosh y - i\,2\sin x \sinh y$

Equate Re & Im-parts



$\cos x \cosh y = \frac{1}{2}$

$\sin x \sinh y = 0 \Rightarrow x = n\pi$ or $y = 0$.

$\underline{y=0}$ $\quad \cos x = \frac{1}{2} \quad x = \pm \frac{\pi}{\cancel{6}\,3} + 2\pi k$

$\underline{x = n\pi}$ $\quad \cos n\pi = (-1)^n$

$\cosh y = (-1)^n/2$ — n must be even since $\cosh y \geq 1$

— $\cosh y = \frac{1}{2}$ has no soln.

Consequently

$\{z \mid \cos z = \tfrac{1}{2}\} = \{ (x,y) = (\tfrac{\pi}{3} + 2\pi k, 0)$

$k \in \mathbb{Z} \}$.