PHY507 / MA507 PS7.1

Problem 1 14.1.13

Solutions to PS 7.

$$f(\theta) = \frac{-ik}{2\pi}\int_0^{2\pi}\int_0^R e^{ik\rho\sin\theta\sin\phi}\,\rho\,d\rho\,d\phi$$

(*) $$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{k^2}{(2\pi)^2}\int_0^{2\pi}\!\!\int\int_0^R\!\!\int e^{ik(\rho\sin\theta\sin\phi - \tilde\rho\sin\theta\sin\tilde\phi)}$$

$$\times\ \rho\tilde\rho\,d\rho\,d\tilde\rho\,d\varphi\,d\tilde\varphi.$$

Use the generating fnc.

$$e^{(ik\rho\sin\theta)\sin\phi} = e^{ik\rho\sin\theta\frac{1}{2i}(e^{i\varphi}-e^{-i\varphi})}$$

$$= e^{(\frac{k\rho}{2}\sin\theta)(t-t^{-1})} \quad \text{with } t = e^{i\varphi}$$

$$= \sum_{n=-\infty}^{\infty} e^{in\varphi} J_n(k\rho\sin\theta)$$

So the integral (*) is

$$\sum_{n,m}\int_0^R \rho\,d\rho\int_0^R\tilde\rho\,d\tilde\rho\; J_n(k\rho\sin\theta)\,J_m(k\tilde\rho\sin\theta)$$

$$\times\int_0^{2\pi}\int_0^{2\pi} e^{im\tilde\varphi}e^{in\varphi}(-1)^m\,d\varphi\,d\tilde\varphi$$

The $(\tilde\varphi, \tilde\varphi)$ integrals are $\delta_{m0}(2\pi)$ so the sum collapses.

$$\frac{d\sigma}{d\Omega} = k^2\left[\int_0^R J_0(k\rho\sin\theta)\,\rho\,d\rho\right]^2$$

(p57.2)

To evaluate this we need

$$\frac{d}{dx}\left(xJ_1\right) = xJ_0$$

This is (14.10) for $n=1$

Then:

$$\int_0^R J_0(k\rho\sin\theta)\rho\, d\rho \qquad u = k\rho\sin\theta$$

$$= \frac{1}{(k\sin\theta)^2}\int_0^{kR\sin\theta} \underline{J_0(u)\, u}\, du \qquad \rho = \frac{u}{k\sin\theta}$$

$$d\rho = \frac{du}{k\sin\theta}$$

$$= \frac{1}{(k\sin\theta)^2}\int_0^{kR\sin\theta} \frac{d}{du}\, uJ_1(u)\, du$$

$$= \frac{1}{(k\sin\theta)^2}\left(kR\sin\theta\right)J_1\left(kR\sin\theta\right)$$

Returning,

$$\frac{d\sigma}{d\Omega} = k^2\frac{R^2}{(k\sin\theta)^2}\left[J_1(kR\sin\theta)\right]^2$$

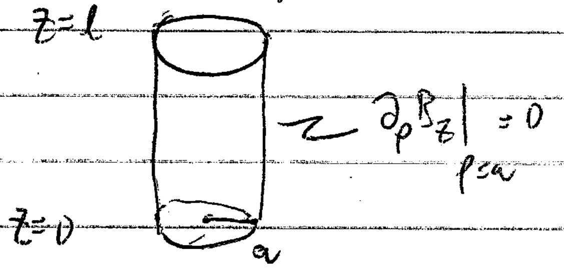
$$= \left(\pi R^2\right)\frac{1}{\pi}\left[\frac{J_1(kR\sin\theta)}{\sin\theta}\right]^2$$

geometric cross-section.

PS7.3

14.1.25 $\Delta B_z + \alpha^2 B_z = 0$ BC $B_z(z=0) = B_z(z=l) = 0$

in a cylinder $\left.\frac{\partial B_z}{\partial \rho}\right|_{\rho=a} = 0$.



Separate: $\left[\frac{1}{\rho}\partial_\rho \rho \partial_\rho + \frac{1}{\rho^2}\partial_\phi^2 + \partial_z^2\right] R(\rho)\Phi(\phi)Z(z)$

$$= -\alpha^2 R(\rho)\Phi(\phi)Z(z)$$

(*) $\frac{1}{\rho R}\partial_\rho \rho \partial_\rho R + \frac{1}{\rho^2 \Phi}\partial_\phi^2 \Phi + \frac{1}{Z}\partial_z^2 Z + \alpha^2 = 0$.

Set $\frac{1}{Z}\partial_z^2 Z = -k^2$ or $Z'' + k^2 Z = 0$.

Soln: $Z(z) = \sin\left(\frac{n\pi z}{l}\right)$ $k^2 = \left(\frac{n\pi}{l}\right)^2$.

(*) $\Rightarrow \frac{1}{R}R'' + \frac{1}{\rho R}R' + \frac{1}{\rho^2 \Phi}\partial_\phi^2 \Phi = \left(\frac{n\pi}{l}\right)^2 - \alpha^2$

$$\frac{\rho^2}{R}R'' + \frac{\rho}{R}R' + \left[\alpha^2 - \left(\frac{n\pi}{l}\right)^2\right]\rho^2 = -\frac{\partial_\phi^2 \Phi}{\Phi} = m^2$$

$\partial_\phi^2 \Phi + m^2 \Phi = 0$ $\Phi(\phi) = e^{im\phi}$ $m \in \mathbb{Z}$

Finally, the radial eqn is:

$$\rho^2 R'' + \rho R' + \left(\left[\alpha^2 - \left(\frac{n\pi}{l}\right)^2\right]\rho^2 - m^2\right) R = 0$$

This is Bessel's eqn:

$$\rho^2 Z''(k\rho) + \rho Z'(k\rho) + (k^2\rho^2 - \nu^2) Z_\nu(k\rho) = 0$$

so

$$J_m\left(\left[\alpha^2 - \left(\frac{n\pi}{l}\right)^2\right]^{\frac{1}{2}} \rho\right) \quad \text{is the soln.}$$

Finally (again!) impose the BC: $\left.\frac{\partial B_z}{\partial \rho}\right|_{\rho=a} = 0$.

$$J_m'\left(\left[\alpha^2 - \left(\frac{n\pi}{l}\right)^2\right]^{\frac{1}{2}} a\right) = 0$$

so $\left[\alpha^2 - \left(\frac{n\pi}{l}\right)^2\right]^{\frac{1}{2}} a = \beta_{mj}$

a zero of $J_m'(x)$: $J_m'(\beta_{mj}) = 0$.

Solving $\boxed{\alpha^2 = \left(\frac{\beta_{mj}}{a}\right)^2 + \left(\frac{n\pi}{l}\right)^2}$

and an elementary soln is

$$J_m\left(\frac{\beta_{mj}}{a}\rho\right) e^{im\phi} \sin\left(\frac{n\pi z}{l}\right)$$

for $j = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$

The frequencies are

$$\omega_{mjn} = \sqrt{\left(\frac{\beta_{mj}}{a}\right)^2 + \left(\frac{n\pi}{l}\right)^2}$$

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Problem 2

14.7.5 Recursion relations:

1. (1) $j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x} j_n(x)$

Recall $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$

The recursion relation for $J_\nu(x)$ is:

(2) $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$ (see 11.1.7)

To obtain (1) set $\nu = n+\frac{1}{2}$ and multiply by $\sqrt{\frac{\pi}{2x}}$ in (2) to get:

$$\sqrt{\frac{\pi}{2x}} J_{n-\frac{1}{2}}(x) + \sqrt{\frac{\pi}{2x}} J_{n+\frac{3}{2}}(x) = \frac{2(n+\frac{1}{2})}{x} \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$$

or $\boxed{j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x} j_n(x)}$

2. We also have $\boxed{(3)\quad n j_{n-1}(x) - (n+1) j_{n+1}(x) = (2n+1) j_n'(x)}$

The related relation for $J_\nu(x)$ is:

(4) $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x)$

Solving (*) for $J_{n+\frac{1}{2}}(x)$:

$$\sqrt{\frac{2x}{\pi}} j_n(x) = J_{n+\frac{1}{2}}(x)$$

So (4) with $\nu = n+\frac{1}{2}$:

$$J_{n-\frac{1}{2}}(x) - J_{n+\frac{3}{2}}(x) = 2J'_{n+\frac{1}{2}}(x)$$

$$\sqrt{\frac{2x}{\pi}}\left[j_{n-1}(x) - j_{n+1}(x)\right] = 2\sqrt{\frac{2}{\pi}}\left[x^{\frac{1}{2}} j_n(x)\right]'$$

$$= 2\sqrt{\frac{2}{\pi}}\left(\frac{1}{2}x^{-\frac{3}{2}} j_n(x) + x^{\frac{1}{2}} j_n'(x)\right)$$

P57.6

$$= \sqrt{\frac{2x}{\pi}} \cdot 2 \cdot \left(\frac{1}{2x} j_n(x) + j_n'(x) \right)$$

so (5) $j_{n-1}(x) - j_{n+1}(x) = \frac{1}{x} j_n(x) + 2 j_n'(x)$

Solve (1) for $j_n(x)$:

$$\frac{1}{x} j_n(x) = \left(\frac{1}{2n+1}\right)\left(j_{n-1}(x) + j_{n+1}(x) \right)$$

and substitute into (5)

$$j_{n-1}(x) - j_{n+1}(x) = \left(\frac{1}{2n+1}\right)\left(j_{n-1}(x) + j_{n+1}(x) \right) + 2 j_n'(x)$$

or $\quad 2 j_n'(x) = j_{n-1}\left(1 - \frac{1}{2n+1}\right) - j_{n+1}(x)\left(1 + \frac{1}{2n+1}\right)$

$$= \frac{1}{2n+1}\left(2n\, j_{n-1}(x) - 2(n+1) j_{n+1}(x) \right)$$

from which (3) follows.

14.7.6 Recursive defn of $j_n(x) = x^n (-1)^n \left(\frac{1}{x}\frac{d}{dx}\right)^n j_0(x)$

This is clearly true for j_0 $(n=0)$ so assume true for n & show for j_{n+1} $(n+1)$:

$j_{n+1}(x)$: $\quad x^{n+1} (-1)^{n+1} \left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} j_0(x)$

$$= x^{n+1} (-1)^{n+1} \left(\frac{1}{x}\frac{d}{dx}\right) \left[\left(\frac{1}{x}\frac{d}{dx}\right)^n j_0(x) \right]$$

$$= x^{n+1} (-1)^{n+1} \left(\frac{1}{x}\frac{d}{dx}\right) \left[\frac{1}{x^n (-1)^n} j_n(x) \right]$$

using the inductive hypothesis for n.

$$= x^n (-1) \frac{d}{dx}\left(x^{-n} j_n(x)\right)$$

$$= x^n (-1) \left\{ -n x^{-n-1} j_n(x) + x^{-n} j_n'(x) \right\}$$

$$= (-1) \left\{ \underbrace{-n x^{-1} j_n(x)}_{\text{from (1)}} + \underbrace{\frac{1}{2n+1}\left(n j_{n-1}(x) - (n+1) j_{n+1}(x)\right)}_{\text{from (3)}} \right\}$$

$$= (-1) \left\{ \frac{-n}{2n+1}\left(j_{n-1}(x) + j_{n+1}(x)\right) + \frac{n}{2n+1} j_{n-1}(x) - \frac{(n+1)}{2n+1} j_{n+1}(x) \right\}$$

$$= (-1) \frac{(-n-n-1)}{2n+1} j_{n+1}(x) = j_{n+1}(x).$$

Problem 3
15.1.1

Legendre Polynomials.
Start with (12.26): $(1-x^2) P_n' = n P_{n-1} - n x P_n$
Differentiate:

$$-2x P_n' + (1-x^2) P_n'' = n P_{n-1}' - n P_n - n x P_n'$$

We need to eliminate P_{n-1}'.
Use (12.25): $P_{n-1}' = -n P_n + x P_n'$
so

$$(n-2)x P_n' + (1-x^2) P_n'' = -n^2 P_n + n x P_n' - n P_n$$

$$-2x P_n' + (1-x^2) P_n'' = -n(n+1) P_n$$

so

$$(1-x^2) P_n'' - 2x P_n' + n(n+1) P_n = 0.$$

This is the Legendre ODE.

15.1.3 (P57-8)

The series form of the Legendre polynomial $P_n(x)$ is

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k} .$$

Compute

$$P_n'(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)! (n-2k)}{2^n k! (n-k)! (n-2k)!} x^{n-2k-1}$$

$$P_n''(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)! (n-2k)(n-2k-1)}{2^n k! (n-k)! (n-2k)!} x^{n-2k-2}$$

When we form $(1-x^2)P_n''(x)$ and $-2xP_n'(x)$

Clearly $-x^2 P_n''$, $-2xP_n'$ & $n(n+1)P_n$ all have terms of the same order:

$$-x^2 P_n'' - 2xP_n' + n(n+1)P_n = \sum_{k=0}^{[n/2]} x^{n-2k} C(k;n) \left[n(n+1) - (n-2k)(n-2k+1) \right]$$

$$= \sum_{k=0}^{[n/2]} x^{n-2k} C(k;n) \left[4kn - (2k)^2 + 2k \right]$$

where $C(k;n) = \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!}$

For the P_n'' term

re-index:
$m = k+1$
$k = m-1$

$$\sum_{k=0}^{[n/2]} x^{n-2k-2} \frac{(2n-2k)! (n-2k)(n-2k-1)}{2^n k! (n-k)! (n-2k)!}$$

$$= \sum_{m=1}^{[n/2]+1} x^{n-2m} \frac{(2n-2m+2)! (n-2m+2)(n-2m+1)}{2^n (m-1)! (n-m+1)! (n-2m+2)!} (-1)^{m-1}$$

$$= + \sum_{m=1}^{[n/2]+1} x^{n-2m} \frac{2m(2n-2m)! (2n-2m+1)}{2^n m! (n-m)! (n-2m)!} (-1)^{m-1}$$

$$= -\sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k} \left[4kn - 4k^2 + 2k \right]$$

By adding these, we get zero.

P57-9

15.1.6 Recall $g(x,t) = \frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{l=0}^{\infty} t^l P_l(x)$

$$\frac{\partial g}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{l=1}^{\infty} l t^{l-1} P_l(x)$$

$$g + 2t\frac{\partial g}{\partial t} = \frac{(1-2xt+t^2) + 2t(x-t)}{(1-2xt+t^2)^{3/2}} = \frac{1-t^2}{(1-2xt+t^2)^{3/2}}$$

$$= \sum_{l=0}^{\infty} (1+2l) t^l P_l(x)$$

So $$\boxed{\frac{1-t^2}{(1-2xt+t^2)^{3/2}} = \sum_{l=0}^{\infty} (2l+1) t^l P_l(x)}$$