PHY/MA 507 (H-1)

Solutions to Test 1

1. i) $f(z) = \dfrac{e^z - 1}{z^2}$ pole at $z_0 = 0$. Analytic on $\mathbb{C}\setminus\{0\}$.

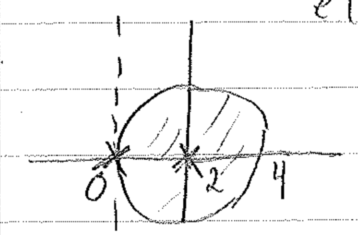
Is $z_0 = 0$ a removable singularity? Compute L.E.

$$e^z - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!}$$

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{k-2}}{k!} = \frac{1}{z} + \frac{1}{2} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$$

So $z_0 = 0$ is a pole of order 1 and $\mathrm{Res}(f,0) = 1$.
Converges on annulus $0 < |z| \leq \infty$

ii) $g(z) = \dfrac{1}{z(z-2)^2}$ 2 poles: one at 0, one at 2



Annular region $0 < |z-2| < 2$ Circle around $z_0 = 2$ radius 2: $A_{0,2}$

Expand in $(z-2)$

$\dfrac{1}{z}$ analytic on disk $0 \leq |z-2| < 2$: $\dfrac{1}{(z-2)+2} = \dfrac{1}{2}\,\dfrac{1}{1+\left(\frac{z-2}{2}\right)}$

$\dfrac{|z-2|}{2} < 1$ so geometric series:

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z-2}{2}\right)^k (-1)^k$$

$\dfrac{1}{(z-2)^2}$ is its own L.E. about $z_0 = 2$.

$$g(z) = \frac{1}{2}\sum_{k=0}^{\infty} (z-2)^{k-2}\left(\frac{-1}{2}\right)^k = \frac{\left(\frac{1}{2}\right)}{(z-2)^2} + \frac{\left(-\frac{1}{4}\right)}{(z-2)} + \frac{1}{8} + \cdots$$

$\mathrm{Res}(g,2) = -1/4$, 2^{nd} order pole.

2\. i) $f(z) = z^2$ analytic so contour integrals are path independent: $\gamma(0) = 0 \quad \gamma(\frac{\pi}{2}) = i$

$$\int_\gamma z^2\,dz = \int_0^i z^2\,dz = \left.\frac{z^3}{3}\right|_0^i = \frac{-i}{3}.$$

10 You can also note $F(z) = z^3/3$ satisfies $F'(z) = z^2$ and use the Fundamental Theorem. (equivalent to above)

15 ii) $J = \int_{|z|=2} \frac{\sin z}{z^2}\,dz \qquad f(z) = \frac{\sin z}{z^2}$ is $\frac{0}{0}$ at origin.

Simple LE $f(z) = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \ldots$

shows a simple pole w/ residue 1.

$J = 2\pi i \,\mathrm{Res}(f, 0) = 2\pi i$

You can also use Cauchy's formula for the derivative:

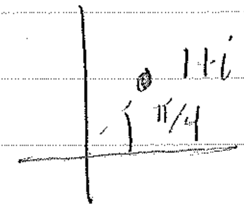
$$J = 2\pi i \left.\frac{d}{dz}\sin z\right|_{z=0} = 2\pi i \cos z\Big|_{z=0} = 2\pi i.$$

(11-3)

3. i) $(1+i)^{2i} = e^{2i \log_{PB}(1+i)}$

$\log_{PB} z = \log|z| + i \arg_{PB} z$

where $-\pi \leq \arg_{PB} z < \pi$



so

$\log_{PB}(1+i) = \frac{1}{2}\log 2 + i\pi/4$

$(1+i)^{2i} = e^{i\log 2 - \pi/2} = e^{-\pi/2}\left(\cos\log 2 + i\sin\log 2\right)$

ii) $v(x,y) = \sinh x \cos y$

$\partial_x v(x,y) = \cosh x \cos y \qquad \partial_y v(x,y) = -\sinh x \sin y$

$\partial_x^2 v(x,y) = \sinh x \cos y \qquad \partial_y^2 v(x,y) = -\sinh x \cos y$

$(\partial_x^2 + \partial_y^2)\, v(x,y) = 0$ so $v(x,y)$ is harmonic

A harmonic conjugate satisfies: $\partial_x u = \partial_y v$

so $u(x,y) = \int_0^x (\partial_y v)(s,y)\,ds$

$= -\int_0^x \sinh s \sin y \, ds = -\cosh x \sin y + C_1$

From the second C-R eqn $\partial_y u = -\partial_x v$

$u(x,y) = \left(-\int_0^y \cos s \, ds\right)\cosh x = -\sin y \cosh x + C_2$

so $u(x,y) = -\cosh x \sin y + C$

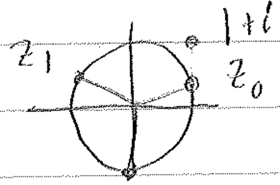
and

$f(z) = -\cosh x \sin y + i \sinh x \cos y + C$

Check CR: $\partial_x u = -\sinh x \sin y \stackrel{?}{=} \partial_y v = -\sinh x \sin y$ ✓

$\partial_y u = -\cosh x \cos y \stackrel{?}{=} -\partial_x v = -\cosh x \cos y$ ✓

$\boxed{H\text{-}4}$

4. 

3 cube roots of i: $z^3 = i = e^{i\pi/2}$

$$z_j = e^{i\left[\frac{\pi}{6}+\frac{2\pi j}{3}\right]}$$

$$z_0 = e^{i\pi/6}, \quad z_1 = e^{i\frac{5\pi}{6}}, \quad z_2 = e^{i\frac{9\pi}{6}} = e^{i\frac{3\pi}{2}}.$$

γ encloses only z_0

$$K = \int_\gamma \frac{1}{z^3-i}\,dz = 2\pi i \,\mathrm{Res}(f, z_0)$$

$$f(z) = \frac{1}{z^3-i} = \frac{1}{(z-z_0)(z-z_1)(z-z_2)}$$

$$\mathrm{Res}(f, z_0) = \frac{1}{(z_0-z_1)(z_0-z_2)}$$

Alternately,

$$f(z) = \frac{g(z)}{h(z)} \quad g(z_0) = 1 \quad h(z_0) = 0 \quad h'(z_0) = 3z_0^2 \neq 0$$

$$\mathrm{Res}(f, z_0) = \frac{1}{3z_0^2} = \frac{1}{3}e^{-\frac{i\pi}{3}} = \frac{1}{3}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = \frac{1}{6}\left(1 - i\sqrt{3}\right)$$

$$K = 2\pi\,\frac{1}{6}(i+\sqrt{3}) = \frac{\pi}{3}(i+\sqrt{3}).$$