MA/PHY 507 Spring 2019
Final Exam - 100 Points
29 April 2019

INSTRUCTIONS: PLEASE WORK ALL FIVE PROBLEMS BELOW. NO BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	25	
2	25	
3	25	
4	25	
TOTAL	100	

Problem 1. (25 points.) The heat equation for a temperature distribution $u(x,t)$ takes the form

$$\frac{\partial u}{\partial t}(x,t) = D\frac{\partial^2 u}{\partial x^2}(x,t) - \lambda u(x,t),$$

for $x \in R$ and $t \geq 0$. The constants D and λ are both positive. Suppose the initial temperature distribution is $u_0(x)$. Use the Fourier transform to write the solution to the initial value problem as

$$u(x,t) = \int_{-\infty}^{\infty} K_t(x,y)u_0(y)\,dy.$$

Write a nice formula for the integral kernel $K_t(x,y)$.
Useful identity: The Fourier transform of e^{-ak^2}, for $a > 0$, is

$$\frac{1}{(2a)^{\frac{1}{2}}}e^{-\frac{x^2}{4a}}.$$

Fourier transform: $\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ikx} u(x,t)\,dx$

PDE becomes

$$\partial_t \hat{u}(k,t) = -Dk^2\hat{u}(k,t) - \lambda\hat{u}(k,t) = -[Dk^2+\lambda]\hat{u}(k,t)$$

Integrate: $\hat{u}(k,t) = Ae^{-(Dk^2+\lambda)t}$

Initial Condition: $\hat{u}(k,t=0) = \hat{u}_0(k) = A$

$$\boxed{\hat{u}(k,t) = e^{-(Dk^2+\lambda)t}\,\hat{u}_0(k)}$$

Take the inverse Fourier transform:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\cdot k}\hat{u}(k,t)\,dk$$

$$= \frac{1}{2\pi}\iint_{\mathbb{R}\mathbb{R}} dk\, e^{i(x-y)k}\, e^{-(Dk^2+\lambda)t}\, u_0(y)\,dy$$

so

$$K_t(x,y) = \frac{e^{-\lambda t}}{2\pi}\int_{\mathbb{R}} dk\, e^{i(x-y)k}\, e^{-Dtk^2}\,dk$$

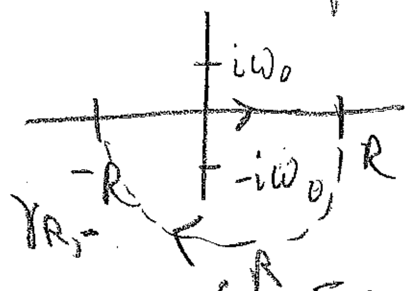
$$= \left(\frac{e^{-\lambda t}}{2\pi}\right)\frac{1}{(2Dt)^{\frac{1}{2}}}e^{-\frac{|x-y|^2}{4Dt}} \quad \text{so} \quad \boxed{K_t(x,y) = \frac{e^{-\lambda t}}{(4\pi Dt)^{\frac{1}{2}}}e^{-\frac{|x-y|^2}{4Dt}}}$$

Problem 2. (25 points.) Compute the integral:

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2+\omega_0^2}\,d\omega, \quad \omega_0 > 0,$$

for all real $t \in R$. Carefully explain your procedure and choice of contour.

$F_t(\omega) = \dfrac{e^{-i\omega t}}{\omega^2+\omega_0^2}$ is meromorphic with simple poles at $\pm i\omega_0$



$$\mathrm{Res}(F_t, \omega_0 i) = \frac{e^{\omega_0 t}}{2i\omega_0}$$

$$\mathrm{Res}(F_t, -i\omega_0) = \frac{e^{-\omega_0 t}}{-2i\omega_0}$$

$$\lim_{R\to\infty} \int_{-R}^{R} F_t(\omega)\,d\omega$$

Close the contour

$\underline{t>0}$ LHP so $e^{-i\omega t} = e^{-i(\mathrm{Re}\,\omega)t}\, e^{(\mathrm{Im}\,\omega)t}$ decays exponentially for $\mathrm{Im}\,\omega < 0$.

$$\oint_{\gamma_{R,-}} F_t(\omega)\,d\omega = -2\pi i\, \mathrm{Res}(F_t, -i\omega_0) = \frac{\pi}{\omega_0} e^{-\omega_0 t}$$

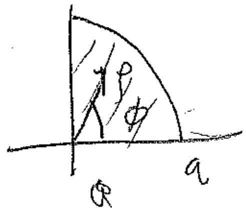
$\underline{t<0}$ UHP so $e^{-i\omega t} = e^{-i(\mathrm{Re}\,\omega)t}\, e^{(\mathrm{Im}\,\omega)t}$ decays exponentially for $\mathrm{Im}\,\omega > 0$

$$\oint_{\gamma_{R,+}} F_t(\omega)\,d\omega = 2\pi i\, \mathrm{Res}(F_t, i\omega_0) = \frac{\pi}{\omega_0} e^{\omega_0 t}$$

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega^2+\omega_0^2}\,d\omega = \frac{\pi}{\omega_0} e^{-\omega_0 |t|},$$

Note: on the semi-circle $Re^{i\theta}$,

$$\int_{\theta=0}^{\pi} |F_t(\omega)| \le \frac{e^{-R\cos\theta |t|}}{|(Re^{i\theta})^2+\omega_0^2|}\, 2\pi R \to 0.$$



Problem 3. (25 points.) Consider the heat equation $\partial_t u = \Delta u$ in a sector of a disk described by $0 \le \rho \le a$ and $0 \le \phi \le \pi/2$ in two-dimensions. Suppose that the edges of the sector are kept at zero temperature.

i. Write down the boundary conditions on $u(\rho, \phi, t)$.

ii. Find the most general **real** solution to the boundary-value problem obtained by separation of variables.

Useful information: Bessel's equation is:

$$x^2 y''(x) + x y'(x) + (\alpha^2 x^2 - \ell^2) y(x) = 0.$$

This ODE has $J_\ell(\alpha x)$ as a regular solution. The Laplacian in polar coordinates is

$$\Delta = \rho^{-1}\partial_\rho \rho \partial_\rho + \rho^{-2}\partial_\phi^2.$$

$u(\rho, \phi, t)$

Basic soln: $\tilde{u}(\rho, \phi, t) = T(t) W(\rho, \phi)$.

$$\frac{T'}{T} = -k^2 = \frac{\Delta W}{W} \Rightarrow T(t) = e^{-k^2} \text{ and } \Delta W = -k^2 W.$$

Radial PDE: $W(\rho, \phi) = R(\rho)\Phi(\phi)$

$$\frac{1}{\rho R}\partial_\rho \rho \partial_\rho R + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = -k^2 \text{ or } \frac{\rho^2}{R}R'' + \frac{\rho}{R}R' + k^2\rho^2 = -\frac{\Phi''}{\Phi}$$

set $\frac{\Phi''}{\Phi} = m^2$, $m \in \mathbb{Z}$, to insure single-valuedness;

$$\Phi(\phi) = A_m \cos m\phi + B_m \sin m\phi$$

Finally: $\rho^2 R'' + \rho R' + [k^2\rho^2 - m^2] R = 0$.

Regular soln. $R(\rho) = J_m(k\rho)$

$$\boxed{\tilde{u}(\rho, \phi, t) = J_m(k\rho)(A_m \cos m\phi + B_m \sin m\phi)e^{-k^2 t}}$$

BC: $\tilde{u}(\rho = a, \phi, t) = 0$

$\tilde{u}(\rho, \phi = 0, t) = 0 = \tilde{u}(\rho, \phi = \frac{\pi}{2}, t)$

$\tilde{u}(\rho, 0, t) = A_m J_m(k\rho) e^{-k^2 t} = 0$ so $A_m = 0$

$\tilde{u}(\rho, \frac{\pi}{2}, t) = B_m \sin(\frac{m\pi}{2}) J_m(k\rho) e^{-k^2 t}$ so $m = 2k$, $k \in \mathbb{N}$.

General Soln: $u(\rho, \phi, t) = \sum_{k=0}^{\infty} C_k \sin(2k\phi) J_{2k}(k\rho) e^{-k^2 t}$

Problem 3 (cont'd)

Finally BC: $\hat{u}(\rho=a,\phi,t)=0 \Longleftrightarrow J_{2k}(ka)=0$

$$\left(\frac{k}{a}\right) \equiv \alpha_{2k,m}; \quad m^{th} \text{ zero of } J_{2k}$$

This introduces a second index m. The general soln is:

$$u(\rho,\phi,t) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} C_{k,m}\sin(2k\phi)\, J_{2k}\left(\frac{\alpha_{2k,m}\rho}{a}\right) e^{-\frac{\alpha_{2k,m}}{a}t}$$

If one has an IC: $u(\rho,\phi,t=0)=u_0(\rho,\phi)$

then:

$$u_0(\rho,\phi) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} C_{k,m}\sin(2k\phi)\, J_{2k}\left(\frac{\alpha_{2k,m}}{a}\rho\right)$$

and the $C_{k,m}$'s are the Fourier-Bessel coefficients of $u_0(\rho,\phi)$.

Problem 4. (25 points.) The generating function $g(x,t)$ for the Hermite polynomials $H_n(x)$ is

$$g(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

Prove the recursion relation for $H_n(x)$ for $n \geq 0$:

$$2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x).$$

Take ∂_t of $g(x,t)$ to avoid derivatives of $H_n(x)$:

$$\partial_t g(x,t) = (-2t+2x)\,g(x,t) = \sum_{n=0}^{\infty} H_n(x)\,\frac{n\,t^{n-1}}{n!}$$

Write the expansion of g and compare coeff. of t^p for all p:

$$(-2t+2x)\,g(x,t) = -2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!} + 2\sum_{n=0}^{\infty} x H_n(x)\frac{t^n}{n!}$$

$$= \sum_{n=1}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

Reindex:

$$-2\sum_{m=1}^{\infty} H_{m-1}(x)\frac{t^m}{(m-1)!} + 2\sum_{m=0}^{\infty} x H_m(x)\frac{t^m}{m!} = \sum_{m=0}^{\infty} H_{m+1}(x)\frac{t^m}{m!}$$

$m = n+1 \qquad m = n-1$

$$\sum_{m=0}^{\infty}\left[-2m\,H_{m-1}(x) + 2x\,H_m(x)\right]\frac{t^m}{m!} = \sum_{m=0}^{\infty} H_{m+1}(x)\frac{t^m}{m!}$$

So for $m \geq 0$: $-2m\,H_{m-1}(x) + 2x\,H_m(x) = H_{m+1}(x)$.