MA/PHY 507

Solutions to PS 10

PS10-1

19.9.20 Clamped String on $[0, L]$

$$\partial_t^2 u = v^2 \partial_x^2 u \text{ with } u(x=0,t) = u(x=L,t) = 0$$

Assume $u(x,t) = \sum b_n(t) \sin \frac{n\pi x}{L}$

Note: $\phi_n(x) = \sin \frac{n\pi x}{L}$ satisfies DBC

and $-\phi_n''(x) = \left(\frac{n\pi}{L}\right)^2 \phi_n(x)$

Substitute u into the WE

$$\partial_t^2 u = \sum_n b_n''(t)\phi_n(x) = -v^2 \sum_n \left(\frac{n\pi}{L}\right)^2 \phi_n(x)\, b_n(t)$$

$$b_n''(t) + v^2\left(\frac{n\pi}{L}\right)^2 b_n(t) = 0.$$

$$b_n(t) = A_n \cos\left(\frac{vn\pi t}{L}\right) + B_n \sin\left(\frac{vn\pi t}{L}\right)$$

Now impose IVC:

$$u(x,0) = f(x) \qquad (\partial_t u)(x,0) = g(x).$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx$$

$$\partial_t u(x,0) = \sum_{n=1}^{\infty} \frac{n\pi v}{L} B_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow B_n = \frac{2}{n\pi v}\int_0^L \sin\left(\frac{n\pi x}{L}\right) g(x)$$

PS10-2

20.2.6 $\lim_{n\to\infty}\int f(x)\,\delta_n(x)\,dx = f(0)$.

Compute $\int f(x)\,\frac{n}{\sqrt{\pi}}\,e^{-n^2x^2}\,dx$. Note: $\int \delta_n(x)\,dx = 1$

$u = nx$

$$= \frac{1}{\sqrt{\pi}}\int f\left(\tfrac{u}{n}\right)e^{-u^2}\,du$$

For n large $f\left(\frac{u}{n}\right) = f(0) + \frac{u}{n}f'(0)$

$+ \frac{1}{2}\left(\frac{u}{n}\right)^2 f''(0) + \cdots$

Leading term:

$$\int f(x)\,\delta_n(x)\,dx = f(0)\,\frac{1}{\sqrt{\pi}}\int e^{-u^2}\,du = f(0)$$

all other terms vanish as $n\to\infty$, like

- $\frac{f'(0)}{\sqrt{\pi}}\,\frac{1}{n}\int e^{-u^2}u\,du = 0$ (symmetry)

- $\frac{f''(0)}{2\sqrt{\pi}}\,\frac{1}{n^2}\int e^{-u^2}u^2\,du \xrightarrow[n\to\infty]{} 0$

This show that:

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(x)\,\delta_n(x)\,dx = f(0) = \int_{\mathbb{R}} f(x)\,\delta(x)\,dx$$

As for the integral identity, note that

PS10-3

$$\hat{\delta}_n(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \delta_n(x) e^{-ikx} dx = \frac{n}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-n^2x^2} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-u^2} e^{-i\frac{k}{n}u} du$$

and $\lim_{n\to\infty} \hat{\delta}_n(k) = \frac{1}{\sqrt{2\pi}} \cdot \int_{\mathbb{R}} e^{-u^2} du = \frac{1}{\sqrt{2\pi}}$

Taking the inverse FT

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} \frac{1}{\sqrt{2\pi}} dk = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} dk$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ikx} dk$$

by change of $k \to -k$

PS 10-4

20.3.6 Diffusion eqn:

$$-D\phi'' + \bar{K}^2 D \phi = Q\,\delta(x)$$

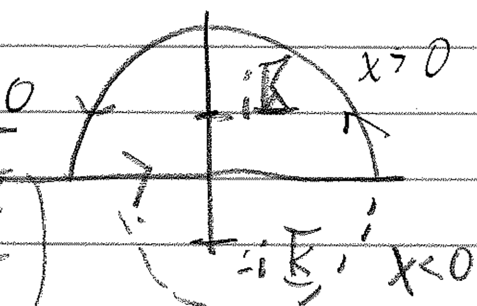
Take FT $\quad \hat{\phi}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} \phi(x)\,dx$

$$D k^2 \hat{\phi} + \bar{K}^2 D \hat{\phi} = \frac{Q}{\sqrt{2\pi}}$$

$$\hat{\phi}(k) = \frac{1}{\sqrt{2\pi}} \frac{Q}{Dk^2 + \bar{K}^2 D} = \frac{1}{\sqrt{2\pi}} \frac{Q}{D} \frac{1}{k^2 + \bar{K}^2}$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int e^{ixk} \left\{ \frac{1}{\sqrt{2\pi}} \left(\frac{1}{k^2 + \bar{K}^2} \right) \right\} \left(\frac{Q}{D} \right)$$

Poles $k = \pm i\bar{K}$, Take $\bar{K} > 0$



$x>0$, $\mathrm{Im}\,k > 0$, UHP

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ixk}\,dk}{(k - i\bar{K})(k + i\bar{K})} = 2\pi i \left(\frac{e^{-x\bar{K}}}{2i\bar{K}} \right) = \frac{\pi}{\bar{K}} e^{-x\bar{K}}$$

$x<0$, $\mathrm{Im}\,k < 0$, LHP

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ixk}\,dk}{(k - i\bar{K})(k + i\bar{K})} \underset{\text{cw}}{=} -2\pi i \left(\frac{e^{x\bar{K}}}{-2i\bar{K}} \right) = \frac{\pi}{\bar{K}} e^{x\bar{K}} \quad \underline{x<0}$$

$$\Rightarrow \boxed{\phi(x) = \frac{Q}{2\bar{K}D} e^{-|x|\bar{K}}}$$

p510.5

9.7.3 $\partial_t \psi = a^2 \partial_x^2 \psi \qquad \psi(x, t=0) = A\,\delta(x)$

$$\psi(x,t) = \int K_t(x,y)\,\psi(y,t=0)\,dy$$

$$= A\,K_t(x,0)$$

Heat kernel at $y=0$.

$$\psi(x,t) = \frac{A}{(4a^2\pi t)^{1/2}} e^{-|x|^2/4a^2 t}$$