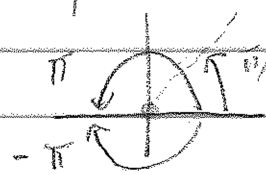
MA/PHY 507   Solns. to PS 2.   PS2-1

1. $\zeta = (1+i)^{(2-i)} = e^{(2-i)\log_{PB}(1+i)}$ by definition

Compute $\log_{PB}(1+i) = \log 2^{\frac{1}{2}} + i\,\arg_{PB}(1+i)$

$$= \frac{1}{2}\log 2 + \frac{i\pi}{4}$$



$$e^{(2-i)\left[\frac{1}{2}\log 2 + i\pi/4\right]}$$

Exponent: $(2-i)\left(\frac{1}{2}\log 2 + i\pi/4\right)$

$$= \left(\log 2 + \pi/4\right) + i\left(\pi/2 - \frac{1}{2}\log 2\right)$$

$$\zeta = 2e^{\pi/4}\left[\cos\left(\pi/2 - \frac{1}{2}\log 2\right) + i\sin\left(\pi/2 - \frac{1}{2}\log 2\right)\right]$$

$$= 2e^{\pi/4}\left[\sin\left(\frac{1}{2}\log 2\right) + i\cos\left(\frac{1}{2}\log 2\right)\right]$$

2. $(2-3i)^4$ $2-3i = \sqrt{13}\,e^{i\theta}$

$$\theta = \arctan(-3/2) \mod (2\pi)$$
$$= -\tan^{-1}(3/2)$$

$$(2-3i)^4 = 13^2 e^{-4i\tan^{-1}(\frac{3}{2})} = 13^2\left[\cos\left(4\tan^{-1}(3/2)\right) - i\sin\left(4\tan^{-1}(3/2)\right)\right]$$

or

$$(2-3i)^2 = 4 - 9 - 12i = -5 - 12i$$

so

$$(2-3i)^4 = (5+12i)^2 = -119 + 120i$$

PS2-2

3. $\sin z = \frac{\sqrt{2}}{2}$ Separate into Re & Im parts

Re: $\sin x \cosh y = \frac{\sqrt{2}}{2}$ (1)

Im: $\cos x \sinh y = 0$ $y=0$ or $x = \frac{(2k+1)\pi}{2}$

If $y=0$ then (1) implies $\sin x = \frac{\sqrt{2}}{2}$ so $x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ and shifts by 2π:

$$x = \frac{\pi}{4} + 2\pi k \quad \text{or} \quad \frac{3\pi}{4} + 2\pi m$$

If $y \neq 0$ and $x = \frac{(2k+1)\pi}{2}$ then (1) implies

$\sin\left(\frac{2k+1}{2}\right)\pi = \pm 1$ so $\pm\cosh y = \frac{\sqrt{2}}{2}$

But $\cosh y \geq 1$ and $\frac{\sqrt{2}}{2} < 1$ so no solution.

So the only solns. are $z = x + i0$ with $x = \begin{cases} \frac{\pi}{4} + 2\pi k \\ \frac{3\pi}{4} + 2\pi m \end{cases}$

11.2.1 $f(z) = x$ $u(x,y) = x$ $v(x,y) = 0$

C-R: $\partial_x u = 1 \neq \partial_y v = 0$ so f isn't analytic

11.2.3 a) $u(x,y) = x^3 - 3xy^2$ Find a harmonic conjugate $v(x,y)$ so $f = u + iv$ is analytic.

Use CR: $\partial_x u = 3x^2 - 3y^2$ $\partial_y u = -6xy$

$\partial_x u = \partial_y v \Rightarrow v(x,y) = \int^y (3x^2 - 3s^2)\, ds = 3x^2 y - y^3 + h(x) + c$

P52-3

$\partial_y u = -\partial_x v \Rightarrow$

$$v(x,y) = \int^x 6sy\,ds = 3x^2 y + g(y) + c$$

Compare: $h(x) = 0$ and $g(y) = -y^3$

$f = u + iv$ with $\boxed{\begin{aligned} u(x,y) &= x^3 - 3xy^2 \\ v(x,y) &= 3x^2 y - y^3 + c. \end{aligned}}$

Check: $f(z) = z^3$

b) $v(x,y) = e^{-y}\sin x \qquad \partial_x v = e^{-y}\cos x \qquad \partial_y v = -e^{-y}\sin x$

$\partial_x u = \partial_y v \Rightarrow$

$$u(x,y) = -\int^x e^{-y}\sin t\,dt = e^{-y}\cos x + g(y) + C$$

$\partial_y u = -\partial_x v \Rightarrow$

$$u(x,y) = -\int^y e^{-s}\cos x\,ds = e^{-y}\cos x + h(x) + c$$

Compare

$\boxed{u(x,y) = e^{-y}\cos x}$

so $f(z) = e^{-y}[\cos x + i\sin x] = e^{-y}e^{ix} = e^{i(x+iy)}$

$= e^{iz}$

11.2.9 b) $f(z) = \frac{1}{z^2+1}$, $\frac{df}{dz}(z) = \frac{-2z}{(1+z^2)^2}$

so f analytic on $\mathbb{C} \setminus \{i, -i\}$.

f) $f(z) = \tan z = \frac{\sin z}{\cos z}$ analytic except where $\cos z = 0$

$\cos(x+iy) = \frac{1}{2}(e^{ix}e^{-y} + e^{-ix}e^{y})$

$= \cos x \cosh y + i \sin x \sinh y$

$\cos x \cosh y = 0$ $x = \frac{(2k+1)\pi}{2}$, any y

$\sin x \sinh y = 0 \Rightarrow y = 0$.

f analytic on $\mathbb{C} \setminus \{\left(\frac{2k+1}{2}\right)\pi, k \in \mathbb{Z}\}$

$\frac{df}{dz} = 1 + \frac{(\sin z)^2}{(\cos z)^2} = \frac{1}{(\cos z)^2} = \sec^2 z$.