PHY/MA 507 (3.1)

Solutions to PS 3

1. $r^2 = x^2+y^2 \qquad \theta = \tan^{-1}(y/x)$

$r = r(x,y)$ and $\theta = \theta(x,y)$. Chain rule:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{x}{r}\right) - \frac{\partial u}{\partial \theta}\left(\frac{y}{r^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\frac{y}{r} + \frac{\partial u}{\partial \theta}\left(\frac{x}{r^2}\right)$$

Similarly:
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{x}{r} - \frac{\partial v}{\partial \theta}\frac{y}{r^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{y}{r} + \frac{\partial v}{\partial \theta}\frac{x}{r^2}$$

We get:

$$\frac{\partial u}{\partial x} = \partial_r u \cos\theta - \partial_\theta v\frac{\sin\theta}{r} = \frac{\partial v}{\partial y} = \partial_r v \sin\theta + \partial_\theta v\frac{\cos\theta}{r}$$

$$\frac{\partial u}{\partial y} = \partial_r u \sin\theta + \partial_\theta u\frac{\cos\theta}{r} = -\frac{\partial v}{\partial x} = -\partial_r v\cos\theta + \partial_\theta v\frac{\sin\theta}{r}$$

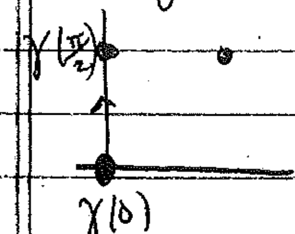
Solve for $\partial_r u = \frac{1}{r}\partial_\theta v$ and $\frac{1}{r}\partial_\theta u = -\partial_r v$.

PHY/MA 507 (3-2)

Solutions to PS 3

1(a) $\int_\gamma z^2\,dz \qquad \gamma(t) = e^{it}\sin^3 t \qquad 0 \le t \le \pi/2$

$\gamma(0) = 0 \qquad \gamma(\pi/2) = i$



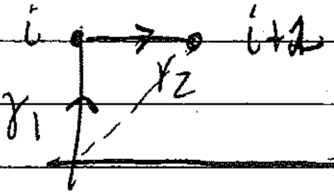
$\frac{1}{3}\frac{d}{dz}z^3 = z^2$ so $\int_\gamma z^2\,dz = \frac{1}{3}z^3\Big|_{\gamma(0)}^{\gamma(\pi/2)}$

$= \frac{1}{3}(i^3 - 0) = -\frac{i}{3}.$

Note We used the fund. th.: if $f = F'$ and γ lies in the region where F is analytic, $\gamma(0) = z_0$ & $\gamma(1) = z_1$ then $\int_\gamma f\,dz = F(z_1) - F(z_0)$

The integral is independent of the path joining z_0 to z_1.

(b) $\int_\gamma y\,dz = \int_\gamma (\mathrm{Im}\, z)\,dz.$



We have to integrate here $\gamma = \gamma_1 \cup \gamma_2$

$\gamma_1(t) = it \quad t \in [0,1] \qquad \gamma_2(t) = i + t \quad t \in [0,2]$

$y|_{\gamma_1} = t \quad dz = i\,dt \qquad y|_{\gamma_2} = 1 \quad dz = dt$

$$\left.\begin{aligned} \int_{\gamma_1} y\,dz &= \int_0^1 t\,i\,dt = i\frac{t^2}{2} = \frac{i}{2} \\ \int_{\gamma_2} y\,dz &= \int_0^2 dt = 2 \end{aligned}\right\} \int_\gamma y\,dz = \frac{1}{2}i + 2 = i/2 + 2$$

This is path dependent. let $\gamma(t) = t(2+i)$, $t \in [0,1]$

$y|_{\gamma(t)} = t \qquad dz = (2+i)\,dt$

$\int_\gamma y\,dz = \int_0^1 (2+i)\,t\,dt = \frac{2+i}{2} = 1 + \frac{i}{2}.$

3.3

Problem 4

• 11.3.4 (a) $I(z)=\int_{\pi(1+i)}^{z} \cos 2\zeta \, d\zeta$. The function $\cos 2\zeta$ has an ~~anti-derivative~~

$F(\zeta)=\frac{1}{2}\sin 2\zeta$

so $F'(\zeta) = \cos 2\zeta$ everywhere

By the fund. theorem:

$$I(z)=\int_{\pi(1+i)}^{z}\cos 2\zeta \, d\zeta = F(z) - F(\pi(1+i))$$

$i\pi \leftarrow \pi(1+i)$

periodicity

$$= \frac{1}{2}\left(\sin 2z - \sin 2\pi(1+i)\right)$$

$\sin z = \sin x \cosh y + i \cos x \sinh y$ so $z = 2\pi + 2\pi i$

$\Rightarrow \sin 2\pi(1+i) = i\sinh 2\pi$

note: $\sin(2\pi + 2\pi i) = \sin(2\pi i)$

and $I(i\pi) = 0$

$$I(z) = \int_{\pi(1+i)}^{z} \cos 2\zeta \, d\zeta = \frac{1}{2}\left(\sin 2z - i\sinh 2\pi\right).$$

Problem 3

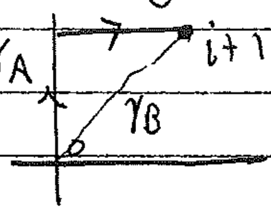
• 11.2.10 a) $f(z) = z^{3/2} = e^{3/2 \log z}$

$$\frac{df}{dz} = e^{3/2\log z} \cdot \frac{3}{2z} = \frac{3}{2}z^{-1}e^{3/2\log z} = \frac{3}{2}z^{1/2}$$

for any branch of the log. The region where f is analytic depends on the branch. For PB, $-\pi < \arg z < \pi$ one excludes the half-axis $(-\infty, 0]$. Then f is analytic on $\mathbb{C} \setminus (-\infty, 0]$. Note that f is the composition of 2 fncs. $e^z = h(z)$ and $\log_{PB} z = g(z)$ so $f(z) = h(g(z))$. Since h is entire, the region of analyticity is determined by g.

Problem 4

(b) 11.3.6. $I = \int_0^{1+i} z^* \, dz$. Path dependent! This is similar to 1(b).



There is no analytic fnc F so $F' = f$.

Let's compute I by 2 paths joining $0 \to 1+i$:

$\underline{\underline{\gamma_A}} = \gamma_1 \cup \gamma_2$ $\quad \gamma_1(t) = it \quad dz = i\,dt \quad z^*|_{\gamma_1} = -it$

$\gamma_2(t) = i + t \quad dz = dt \quad z^*|_{\gamma_2} = t - i$

$$\int_{\gamma_A} z^* dz = \int_0^1 (-it)(i\,dt) + \int_0^1 (t-i)\,dt$$

$$= \frac{1}{2} + \frac{1}{2} - i = 1 - i.$$

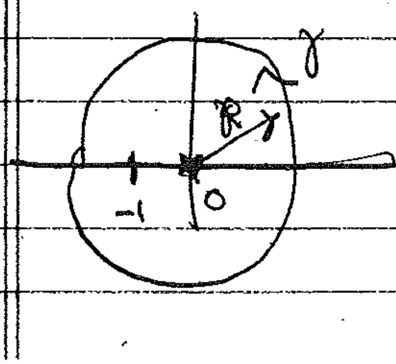
$\underline{\underline{\gamma_B}}(t) = t(1+i) \quad dz = dt\,(1+i) \quad z^*|_{\gamma_B} = t(1-i)$

$$\int_{\gamma_B} z^* dz = \int_0^1 t(1-i)(1+i)\,dt = 2 \cdot \frac{1}{2} = 1.$$

These 2 are different so I isn't path independ[ent]

(c) 11.3.7. $\displaystyle\int_{|z| = R > 1} \frac{dz}{z^2 + z}$ $\quad f(z) = \dfrac{1}{z(z+1)}$ has 2 poles at $0, -1$ so f isn't analytic inside γ.

Both lie inside γ. $f(z) = \dfrac{1}{z} - \dfrac{1}{z+1}$



$$\int_\gamma f(z)\,dz = \int_\gamma \frac{dz}{z} - \int_\gamma \frac{dz}{z+1}$$

$$= 2\pi i - 2\pi i = 0. \quad \blacksquare$$