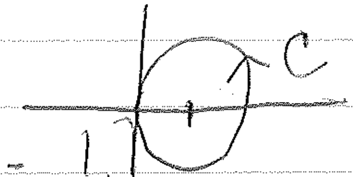
PHY/MA 507 (PS 4-1)

Solutions to P.S. 4

11.4.2 $\displaystyle\oint_C \frac{dz}{z^2-1}$ $C = \{ z \mid |z-1| = 1 \}$



$$\frac{1}{z^2-1} = -\frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z-1}\right]$$

$$\oint_C \frac{dz}{z^2-1} = \frac{1}{2}\oint_C \frac{dz}{z-1} - \cancel{\oint_C \frac{dz}{z+1}}^{\,0}$$

since $\frac{1}{z+1}$ analytic on & inside C

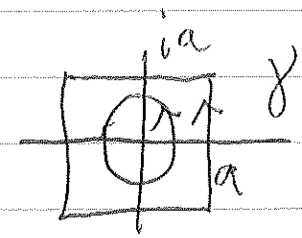
$$= \frac{1}{2}\cdot 2\pi i$$

$$= \pi i$$

You can also write $\displaystyle\oint \frac{\left(\frac{1}{z+1}\right)}{z-1}dz = 2\pi i \cdot \frac{1}{2}$ since $\frac{1}{z+1}$ is analytic on & inside C.

You can also write $\displaystyle\oint \frac{f(z)}{z-1}dz$ with $f(z) = \frac{1}{z+1}$ and use Cauchy's Integral Formula.

11.4.6 $\displaystyle J = \oint_C \frac{e^{iz}}{z^3}dz$



Deformation says any simple closed contour about 0 is equivalent to the square.

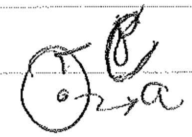
$$f^{(k)}(0) = \frac{k!}{2\pi i}\oint_C \frac{f(z)}{z^{k+1}}dz \qquad k = 2 \quad f(z) = e^{iz} \text{ entire}$$

$$z_0 = 0$$

$$J = \frac{2\pi i}{2!}\frac{d^2}{dz^2}e^{iz}\Big|_{z=0} = \pi i \cdot i^2 e^{iz}\Big|_{z=0}$$

$$= -\pi i$$

11.4.7 $\displaystyle\oint_C \frac{\sin^2 z - z^2}{(z-a)^3}dz$



$$= \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left(\sin^2 z - z^2\right)\Big|_{z=a}$$

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$$= \pi i \frac{d}{dz}\left(2\sin z\cos z - 2z\right)\Big|_{z=a}$$

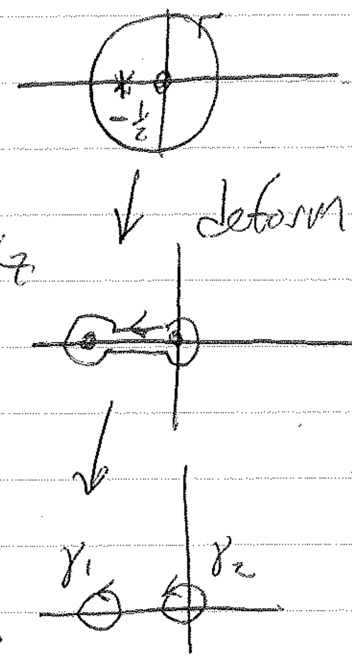
$$= 2\pi i\left[\cos^2 z - \sin^2 z - 1\right]\Big|_{z=a}$$

$$= 2\pi i\left[1 - 2\sin^2 z - 1\right]\Big|_{z=a}$$

$$= -4\pi i \sin^2 a.$$

11.4.9. $K \equiv \oint_{|z|=1} \frac{f(z)}{z(2z+1)^2}\, dz$

Assume f analytic



$$\oint_{|z|=1} \frac{f(z)\,dz}{z(2z+1)^2} = \oint_{|z+\frac{1}{2}|=\varepsilon} \frac{[f(z)/z]\,dz}{(2z+1)^2}$$

$$+ \oint_{|z|=\varepsilon'} \frac{f(z)/(2z+1)^2}{z}\,dz$$

and use Cauchy for each:

$g(z) = \frac{f(z)}{z}$ analytic on & inside $|z+\frac{1}{2}|=\varepsilon$.

$$\oint_{|z+\frac{1}{2}|=\varepsilon} \frac{g(z)}{4(z+\frac{1}{2})^2}\,dz = \frac{1}{4}\left(\frac{2\pi i}{1!}\right)\frac{d}{dz}g(z)\Big|_{z=-\frac{1}{2}}$$

$$= \left(\frac{\pi i}{2}\right)\left[\frac{f'(z)}{z} - \frac{f(z)}{z^2}\right]\Big|_{z=-\frac{1}{2}}$$

$$= \left(\frac{\pi i}{2}\right)\left[-2f'(-\tfrac{1}{2}) - 4f(-\tfrac{1}{2})\right]$$

$h(z) = \dfrac{f(z)}{(2z+1)^2}$ analytic on & inside $|z| = \varepsilon'$

$$\oint_{|z|=\varepsilon'} \frac{h(z)}{z}\,dz = 2\pi i\, h(0) = 2\pi i\, f(0)$$

so

$$K = -\pi i \left[f'(-\tfrac{1}{2}) + 2f(-\tfrac{1}{2})\right] + 2\pi i\, f(0).$$

You can also use partial fractions:

$$\frac{1}{z(2z+1)^2} = \frac{A}{z} + \frac{B}{2z+1} + \frac{C}{(2z+1)^2}$$

$$1 = A(4z^2+4z+1) + Bz(2z+1) + Cz$$
$$= (4A+2B)z^2 + (4A+B+C)z + A$$

$$\left.\begin{array}{l} 4A+2B=0 \\ 4A+B+C=0 \\ A=1 \end{array}\right\} \quad \begin{array}{l} A=1 \\ B=-2 \\ C=-2 \end{array}$$

$$\oint_{|z|=1} \frac{f(z)}{z(2z+1)^2}\,dz = \oint \frac{f(z)}{z}\,dz - \frac{2}{2}\oint \frac{f(z)}{(z+\frac{1}{2})}\,dz$$
$$- \frac{2}{2^2}\oint \frac{f(z)}{(z+\frac{1}{2})^2}\,dz$$
$$= 2\pi i\, f(0) - 2\pi i\, f(-\tfrac{1}{2}) - \frac{(2\pi i)}{2}\left[\frac{1}{2\pi i}\oint \frac{f(z)}{(z+\frac{1}{2})^2}\,dz\right]$$
$$= 2\pi i\left(f(0) - f(-\tfrac{1}{2})\right) - f'(-\tfrac{1}{2})\,\pi i.$$

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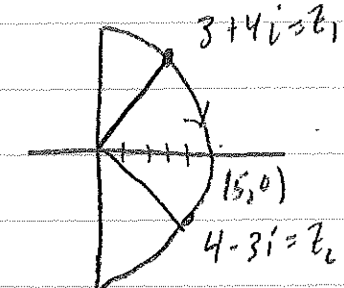
11.3.3 $$\int_{3+4i}^{4-3i} \left(4z^2 - 3iz\right) dz = F(4-3i) - F(3+4i)$$

where $F(z) = \frac{4}{3}z^3 - \frac{2}{3}iz^2$

is an anti derivative

Note $f(z) = 4z^2 - 3iz$ is entire.



(1) $z = 5e^{i\theta}$ with $\theta \in [\theta_1, \theta_2]$

$$\int_{z_1}^{z_2} f(z)\,dz = \int_{\theta_1}^{\theta_2} \left(100e^{2i\theta} - 15i\,e^{i\theta}\right)\left(i5e^{i\theta}\,d\theta\right)$$

$$= \frac{500i}{3i}\left(e^{3i\theta_2} - e^{3i\theta_1}\right) + \frac{75}{2i}\left(e^{2i\theta_2} - e^{2i\theta_1}\right)$$

$$= \frac{76}{3} - i\frac{707}{3}$$

(2) Easiest: $\gamma(t) = t(4-3i) + (1-t)(3+4i)$, $t \in [0,1]$

$\gamma'(t) = 4 - (3+4i) = (1-4i)$

$$\int_{3+4i}^{4-3i} f(z)\,dz = \int_0^1 f(\gamma(t))\,\gamma'(t)\,dt = \int_0^1 \left(4\gamma(t)^2 - 3i\gamma(t)\right)(1-4i)\,dt$$

write $\gamma(t) = (3+t) + i(4-7t)$

One gets the same answer after integration