MA/PHY 507
Solutions to PS 5

PS5-1

1. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{z^2}{2^2}\right)^n - 1$ written like this we see the series is a geometric series with ratio $z^2/2^2 = r$

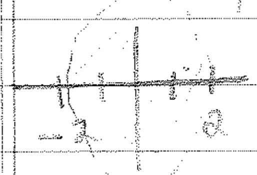
Convergence requires $|r|<1$ so $|z|^2 < 2^2$
$\Rightarrow |z| < 2 = R$

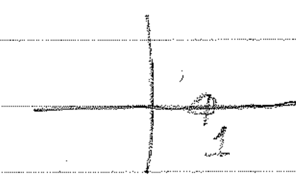
You can also use the ratio test directly:

$$\left|\left(\frac{z^2}{2^2}\right)^{n+1} \Big/ \left(\frac{z^2}{2^2}\right)^{n}\right| = \left|\frac{z}{2}\right|^2 < 1$$

$R = 2$. Sum the series: $\displaystyle\sum r^n = \frac{1}{1-r} = \frac{1}{1 - z^2/2^2}$

$f(z) = \dfrac{z^2}{4-z^2}$ This is analytic for $|z|<2$ $\quad = \dfrac{4}{4-z^2} - 1 = \dfrac{z^2}{4-z^2}$





fnc. is analytic on $\mathbb{C}\setminus\{1\}$
$r_1 = 0,\ r_2 = \infty$

2. (11.5.7) $\dfrac{e^z z}{z-1}$ about $z_0 = 1$

$$e^z z = (e^{z-1}e)(z-1+1) = e\left(\sum_{k=0}^{\infty}\frac{(z-1)^k}{k!}\right)(z-1) + e e^{z-1}$$

$$\frac{e^z z}{z-1} = e\left(\sum_{k=0}^{\infty}\frac{(z-1)^k}{k!}\right) + e\sum_{k=0}^{\infty}\frac{(z-1)^{k-1}}{k!} = \frac{e}{z-1} + \sum_{k=0}^{\infty}\frac{e}{k!}\left(\frac{k+2}{k+1}\right)(z-1)^k$$

$\text{Res}(f,1) = e$. Simple pole.

(11.5.8) $(z-1)e^{\frac{1}{z}}$, $z_0 = 0$

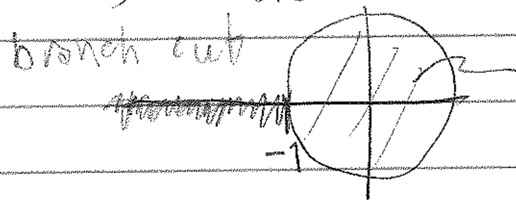
$$= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{z^{k-1}} - \frac{1}{z^k}\right) = z - \sum_{m=0}^{\infty}\frac{m}{(m+1)!}\frac{1}{z^m}$$

$\text{Res}(f,1) = -1$ essential singularity.

PS5-2

2. (1.1.5) $\log_{PB}(1+z)$ is analytic on $\mathbb{C} \setminus (-\infty,-1]$

Taylor exp. about $z_0 = 0$.



analytic in $|z|<1$.

$$\log_{PB}(1) = 0 \quad \text{and} \quad \frac{d}{dz}\log_{PB}(1+z) = \frac{1}{1+z}$$

$$= \sum_{k=0}^{\infty} (-z)^k \quad \text{for } |z|<1.$$

$$\Rightarrow \quad \log_{PB}(1+z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{k+1}}{k+1} \quad \text{since}$$

$\log_{PB}(1) = 0$.

You can also use: $\quad f^{(k)}(z) = \dfrac{(-1)^{k-1}(k-1)!}{(1+z)^k}$

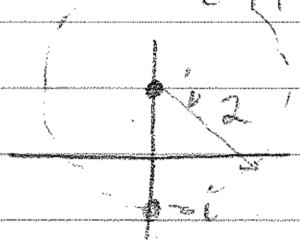
so $f^{(k)}(0) = (-1)^{k-1}(k-1)!$

$$\Rightarrow \quad \log_{PB}(1+z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} z^{k+1}$$

p55-3

3. $f(z) = \dfrac{z}{z^2+1} \Rightarrow \dfrac{z}{(z-i)(z+i)}$ 2 simple poles at $\pm i$



The LE about $z_0 = i$ has $R = 2$

$$\frac{z-i+i}{(z-i)(z-i+2i)} = \frac{1}{2i\left(1+\frac{z-i}{2i}\right)} + \frac{1}{2(z-i)}\frac{1}{\left(1+\frac{z-i}{2i}\right)}$$

Note if $0 < |z-i| < 2$, $\left|\dfrac{z-i}{2i}\right| = \dfrac{|z-i|}{2} < 1$

so $$\frac{1}{1+\left(\frac{z-i}{2i}\right)} = \sum_{k=0}^{\infty} \frac{(-1)^k (z-i)^k}{(2i)^k} = \sum_{k=0}^{\infty} \left(\frac{i}{2}\right)^k (z-i)^k$$

$$f(z) = \frac{1}{2i}\sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^k (z-i)^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^k (z-i)^{k-1}$$

$$= \left(\frac{\left(\frac{1}{2}\right)}{z-i}\right) + \left(-\frac{1}{2}\right)\sum_{k=0}^{\infty}\left(\frac{i}{2}\right)^{k+1}(z-i)^k$$

$\mathrm{Res}(f, i) = \frac{1}{2}$. Simple pole.

You can also write $$f(z) = \frac{1}{2}\left[\frac{1}{z+i} + \frac{1}{z-i}\right]$$

$$= \frac{1}{2}\left(\frac{1}{z-i}\right) + \underbrace{\frac{1}{2}\left(\frac{1}{2i}\right)\left(\frac{1}{1+\frac{z-i}{2i}}\right)}_{\text{analytic}}$$