MA/PHY 507

PS6-1

Solutions to PS 6

11.7.1 a) $g(z) = \frac{1}{z^2+a^2}$, $a>0$. $g(z) = \frac{1}{(z-ia)(z+ia)}$ simple poles at $\pm ia$.

$$\text{Res}(g, ia) = \lim_{z\to ia} (z-ia)g(z) = \frac{1}{2ia} = \frac{-i}{2a}$$

$$\text{Res}(g, -ia) = \lim_{z\to -ia} (z+ia)g(z) = \frac{1}{-2ia} = \frac{i}{2a}.$$

b) $f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z-ia)^2(z+ia)^2}$ double poles at $z_0 = \pm ia$

Note that: z near ia, $g(z) = (z+ia)^{-2}$ analytic and $g(ia) \neq 0$. $f(z) = \frac{g(z)}{(z-ia)^2}$

$$g(z) = g(ia) + g'(ia)(z-ia) + (\text{higher order terms})$$

so
$$f(z) = \frac{1}{(z-ia)^2}\left[g(ai) + g'(ia)(z-ia) + \dots\right]$$
$$= \frac{g(ia)}{(z-ia)^2} + \frac{g'(ia)}{z-ia} + \left(\text{analytic terms}\right)$$

$$\text{Res}(f, ia) = g'(ia) = -2(2ia)^{-3} = \frac{-i}{4a^3}$$

Similarly, for the double pole at $-ia$:

$$f(z) = \frac{h(-ia)}{(z+ia)^2} + \frac{h'(-ia)}{z+ia} + (\text{analytic})$$

where $h(z) = \frac{1}{(z-ia)^2}$ so $h'(z) = -2(z-ia)^{-3}$

$$\text{Res}(f, -ia) = -2(-2ia)^{-3} = \frac{i}{4a^3}$$

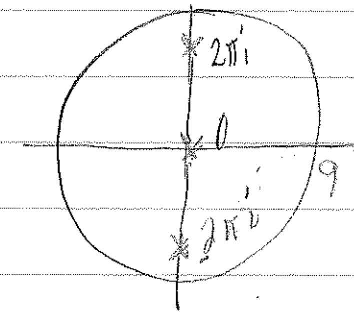
PS6.2

f) $f(z) = \dfrac{ze^{iz}}{z^2-a^2} = \dfrac{ze^{iz}}{(z-a)(z+a)}$ simple poles at $\pm a$.

$\mathrm{Res}(f,a) = \lim_{z\to a} (z-a)f(z) = \dfrac{e^{ia}}{2}$

$\mathrm{Res}(f,-a) = \lim_{z\to -a} (z+a)f(z) = \dfrac{-e^{-ia}}{-2} = \dfrac{e^{-ia}}{2}$

Problem 2 a) $J = \oint_{|z|=9} \dfrac{dz}{e^z-1}$



Roots: $e^z - 1 = 0$. Solve $e^x(\cos y + i\sin y) = 1$

$\begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0 \end{cases}$

If $e^x \sin y = 0 \Rightarrow \sin y = 0 \Rightarrow y = n\pi,\ n\in\mathbb{Z}$

Then $e^x\cos(n\pi) = 1$ iff $n = 2m,\ m\in\mathbb{Z}$ and $x=0$.

Zeros: $(x=0,\ y=2m\pi)$ so $z = 2m\pi i$

Inside $|z|=9$, only $m=-1,0,1$ (3 poles)

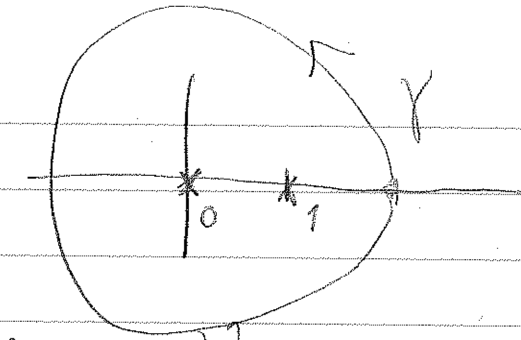
$J = 2\pi i \sum_{m=-1}^{1} \mathrm{Res}(f, z_m)$

$\mathrm{Res}(f,z_m) = \dfrac{g(z_m)}{h'(z_m)} = \dfrac{1}{e^{z_m}} = \dfrac{1}{e^{2\pi m i}} = e^{-2\pi m i} = 1$

$\Rightarrow J = 2\pi i\left(e^{2\pi i} + 1 + e^{-2\pi i}\right) = 6\pi i.$

P56-3

b) $K = \displaystyle\int_{|z|=2} \frac{5z-2}{z(z-1)}\,dz$



$= 2\pi i \left[\mathrm{Res}(f,0) + \mathrm{Res}(f,1) \right]$

$= 2\pi i \left[\left(\frac{-2}{-1}\right) + 3 \right] = \underline{\underline{10\pi i}}$.