MAIANY 507
Solutions to PS 7

PS7-1

9.4.2. $* \quad \nabla^2\psi + k^2\psi = 0 \qquad (\rho, \phi, z)$

$$\nabla^2 = \frac{1}{\rho}\partial_\rho \rho \partial_\rho + \frac{1}{\rho^2}\partial_\phi^2 + \partial_z^2$$

(*) Separable in these coordinates. Now consider the operator: $\nabla^2 + f(\rho) + \frac{1}{\rho^2}g(\varphi) + h(z) + k^2$

Show this is separable: let $\psi(\rho,\phi,z) = R(\rho)\Phi(\phi)Z(z)$
Substitute:

$$\left(\frac{1}{\rho}\partial_\rho \rho \partial_\rho R\right)(\Phi Z) + \left(\frac{1}{\rho^2}\Phi''\right)RZ + R\Phi Z''$$

$$+ (f(\rho)R)\Phi Z + \frac{1}{\rho^2}(g(\varphi)\Phi)RZ + R\Phi(h(z)Z)$$

$$+ k^2 R\Phi Z = 0.$$

Divide by $R\Phi Z$:

$$\frac{1}{R\rho}\partial_\rho \rho \partial_\rho R + \frac{1}{\rho^2}\left(\frac{\Phi''}{\Phi} + g(\varphi)\right) + \left(h(z) + \frac{Z''}{Z}\right) = -k^2$$

so

$$\frac{1}{\rho^2}\left[\left(k^2\rho^2 + \frac{\rho}{R}\partial_\rho \rho \partial_\rho R\right) + \left(\frac{\Phi''}{\Phi} + g(\varphi)\right)\right] = -\left(h(z) + \frac{Z''}{Z}\right)$$

Introduce a separation constant β^2:

$$(1) \quad Z'' + h(z)Z + \beta^2 Z = 0$$

P57-2

and:

$$\frac{\rho}{R}\partial_\rho \rho \partial_\rho R + (k^2-\beta^2)\rho^2 = -\left(\Phi'' + g(\varphi)\Phi\right)\frac{1}{\Phi}$$

and a second separation const. γ^2 to get

$$② \quad \rho^2 R'' + \rho R' + \left((k^2-\beta^2)\rho^2 - \gamma^2\right)R = 0$$

$$③ \quad \Phi'' + g(\varphi)\Phi + \gamma^2\Phi = 0 .$$

① - ② - ③ are 3 ODEs in ρ, φ, z respectively.

(PS7.3)

9.4.5 $\quad -\alpha \Delta \psi = E\psi \qquad \alpha = \hbar^2/2m$ in cube $[0,a]\times[0,b]\times[0,c]$

Rectangular symmetry

- Dirichlet boundary conditions

$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \qquad$ Let $\psi(x,y,z) = X(x)Y(y)Z(z)$

$$-\alpha\left\{YZ\,\partial_x^2 X + XZ\,\partial_y^2 Y + XY\,\partial_z^2 Z\right\} = E\,XYZ$$

$$(*) \quad \frac{1}{X}\partial_x^2 X + \frac{1}{Y}\partial_y^2 Y + \frac{1}{Z}\partial_z^2 Z = -\frac{E}{\alpha}$$

Separation variables: $\dfrac{1}{X}\partial_x^2 X = -k_1^2 \qquad X(0) = X(a) = 0$

$$\partial_x^2 X + k_1^2 X = 0 \qquad X(x) = A_1\cos k_1 x + B_1 \sin k_1 x$$

DBC $\quad X(0) = A_1 = 0$

$X(a) = B_1 \sin k_1 a = 0 \Rightarrow k_1 a = n_1\pi \quad$ so $\boxed{k_1 = \dfrac{n_1\pi}{a}}$

$n_1 \in 1, 2, 3, 4, \ldots$

Return to PDE $(*)$

$$\frac{1}{Y}\partial_y^2 Y + \frac{1}{Z}\partial_z^2 Z = -\frac{E}{\alpha} + \left(\frac{n_1\pi}{a}\right)^2$$

Set $\dfrac{1}{Y}\partial_y^2 Y = -k_2^2$. The same calculation shows

$Y(y) = \sin\left(\dfrac{n_2\pi y}{b}\right)$ & $\boxed{k_2 = \dfrac{n_2\pi}{b}}$

Finally: $\dfrac{1}{Z}\partial_z^2 Z = -\dfrac{E}{\alpha} + \left(\dfrac{n_1\pi}{a}\right)^2 + \left(\dfrac{n_2\pi}{b}\right)^2$

set $\dfrac{1}{Z}\partial_z^2 Z = -k_3^2$ so $Z(z) = \sin\left(\dfrac{n_3\pi z}{c}\right)$ & $\boxed{k_3 = \dfrac{n_3\pi}{c}}$

Whence $E_{\underline{n}} = \dfrac{(\pi\hbar)^2}{2m}\left[\left(\dfrac{n_1}{a}\right)^2 + \left(\dfrac{n_2}{b}\right)^2 + \left(\dfrac{n_3}{c}\right)^2\right]$

The ground state has $n_1 = n_2 = n_3 = 1$.

(PS7-4)

9.4.7 $-\alpha\psi'' + \frac{1}{2}kx^2\psi = E\psi$ harmonic oscillator

$\alpha = \hbar^2/2m$

$$\psi'' + \frac{E}{\alpha}\psi - \frac{kx^2}{2\alpha}\psi = 0 \quad \text{or} \quad \psi'' + \frac{2mE}{\hbar^2}\psi - \frac{mk}{\hbar^2}x\psi = 0$$

Set $a = \left(\frac{mk}{\hbar^2}\right)^{\frac{1}{4}}$ and $\lambda = \frac{2E}{\hbar}\sqrt{\frac{m}{k}}$ and $\xi = ax$

Then $\phi(\xi) = \psi(\xi/a)$, $\frac{d\phi}{d\xi} = \frac{1}{a}\psi'(x)$ or $\psi'(x) = a\frac{d\phi}{d\xi}$

so

$$a^2\frac{d^2\phi(\xi)}{d\xi^2} + \left[\lambda a^2 - a^4\left(\frac{\xi}{a}\right)^2\right]\phi(\xi) = 0$$

so $$\frac{d^2\phi}{d\xi^2}(\xi) + (\lambda - \xi^2)\phi(\xi) = 0 \quad (*)$$

since $$\frac{2mE}{\hbar^2} = \frac{2E}{\hbar}\sqrt{\frac{m}{k}}\left(\frac{mk}{\hbar^2}\right)^{\frac{1}{2}} = \lambda a^2$$

and $$\frac{mk}{\hbar^2} = a^4$$

To reduce this ONE set $\phi(\xi) = y(\xi)e^{-\xi^2/2}$

$$\phi'(\xi) = [y'(\xi) - \xi y(\xi)]e^{-\xi^2/2}$$

$$\phi''(\xi) = [y''(\xi) - y(\xi) - \xi y'(\xi)]e^{-\xi^2/2} + [y'(\xi) - \xi y(\xi)](-\xi e^{-\xi^2/2})$$

$$= [y''(\xi) - 2\xi y'(\xi) - (1-\xi^2)y(\xi)]e^{-\xi^2/2}$$

(*) becomes $\boxed{y''(\xi) - 2\xi y'(\xi) + (\lambda - 1)y(\xi) = 0}$

Hermite eqn.