PHY/MA 507

PS8-1

Solutions to PS8

1. 13.1.1 $\Gamma(z+1)=\int_0^\infty e^{-t}t^z\,dt$ as $t\geq 0$, $t^z=e^{z\log t}$

so $\frac{d}{dt}t^z=\frac{z}{t}t^z=zt^{z-1}$

Integrate-by-parts:

$$\int_0^\infty e^{-t}t^z\,dt=\lim_{M\to\infty}\int_0^M e^{-t}t^z\,dt$$

so

$$\int_0^M e^{-t}t^z\,dt=-t^ze^{-t}\Big|_0^M+\int_0^M zt^{z-1}e^{-t}\,dt$$

$u=t^z \quad du=zt^{z-1}\,dt$

$dv=e^{-t} \quad v=-e^{-t}$

Because $\operatorname{Re} z>0$ $|t^z|=t^{\operatorname{Re} z}$ and $\lim_{M\to\infty}M^{\operatorname{Re} z}e^{-M}=0$

as well as $t^ze^{-t}\Big|_{t=0}=0$.

So

$$\Gamma(z+1)=\int_0^\infty e^{-t}t^z\,dt=z\lim_{M\to\infty}\int_0^M t^{z-1}e^{-t}\,dt=z\,\Gamma(z).$$

13.1.2.

a) $$f_n(s)=\frac{(n+1)(n+2)\cdots(n+2s-1)(n+2s)}{[2\cdot4\cdot6\cdots(2s-2)\,2s]\cdot[(2n+3)\cdot(2n+5)\cdots(2n+2s+1)]}$$

$$=\frac{(n+2s)!}{n!}\cdot\frac{1}{(2s)!!}\cdot\frac{(2n+1)!!}{(2n+2s+1)!!}$$

using the definitions of the double factorial

We have:

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$$\left.\begin{aligned} (2n)!! &= 2^n n! \\ (2n+1)!! &= \frac{(2n+1)!}{2^n n!} \end{aligned}\right\} \text{ for } n \in \mathbb{N}$$

Applying this:

$$C_n(s) = \frac{(n+2s)!}{n!} \cdot \frac{1}{2^s s!} \cdot \frac{(2n+1)!}{2^n n!} \cdot \frac{2^{n+s}(n+s)!}{(2(n+s)+1)!}$$

$$\boxed{C_n(s) = \frac{(n+2s)!}{s!\,n!\,n!} \cdot \frac{(2n+1)!\,(n+s)!}{(2(n+s)+1)!}}$$

b. Pochhammer symbol : $(a)_n = a(a+1)(a+2)\dots(a+n-1),\ n \in \mathbb{N}$

$(a)_0 = 1$

and $(a)_n = \dfrac{(a+n-1)!}{(a-1)!}$

In terms of this, we compute

- $\dfrac{(n+s)!}{s!} = (s+1)_n$
- $\dfrac{(2s+n)!}{n!} = (n+1)_{2s}$
- $\dfrac{(2(n+s)+1)!}{(2n+1)!} = (2n+2)_{2s}$ and $(1)_n = n!$

so that

$$\boxed{C_n(s) = \frac{(s+1)_n\,(n+1)_{2s}}{(1)_n\,(2n+2)_{2s}}}$$

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4.1.3 $g(x,t) = e^{\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$

$$g(-x,-t) = \sum_{n=-\infty}^{\infty} t^n (-1)^n J_n(-x)$$

$$= e^{\left(\frac{-x}{2}\right)\left(-t+\frac{1}{t}\right)} = g(x,t)$$

Comparing coefficients of $t^n$ we find:

$$(-1)^n J_n(-x) = J_n(x)$$

4.1.5 Expansion: $e^{i\rho\cos\varphi} = \sum i^m J_m(\rho) e^{im\phi}$

$$\cos\varphi = \frac{1}{2}\left(u + \frac{1}{u}\right) \qquad u = e^{i\varphi}$$

$$i\rho\cos\varphi = \frac{\rho}{2}\left[iu - \frac{1}{iu}\right] \qquad \text{let } t = iu = ie^{i\varphi}$$

$$e^{i\rho\cos\varphi} = e^{\left(\frac{\rho}{2}\right)\left(t-\frac{1}{t}\right)} = \sum_{m=-\infty}^{\infty} J_m(\rho)\,(ie^{i\varphi})^m$$

$$= \sum_{m=-\infty}^{\infty} i^m J_m(\rho)\, e^{im\varphi}$$

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3. 14.1.13

$$f(\theta) = \frac{-ik}{2\pi} \int_0^{2\pi} d\phi \int_0^R \rho\, d\rho\, e^{ik\rho \sin\theta \sin\phi}$$

diff. X-sec:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 \quad \text{compute this}$$

Use the integral rep.: $J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \cos(x \sin\theta - n\theta)$

and the related identity

$$0 = \int_0^{2\pi} d\theta \sin(x \sin\theta - n\theta)$$

Compute: $f(\theta) = \frac{-ik}{2\pi} \int_0 \rho\, d\rho \int_0 \left[\cos\left[(k\rho\sin\theta)\cdot \sin\phi\right] d\phi\right.$

since $\int_0^{2\pi} \sin\left[(k\rho\sin\theta)\sin\phi\right] d\phi = 0$

so

$$f(\theta) = -ik \int_0^R \rho\, d\rho\, J_0(k\rho\sin\theta)$$

Using the identity

$$x J_0(x) = \left[x J_1(x)\right]'$$

so evaluating the $\rho$-integral we get:

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$$\int_0^R d\rho\, \rho J_0(k\rho \sin\theta) = \int_0^{kR\sin\theta} \frac{1}{(k\sin\theta)^2}\, u J_0(u)\, du$$

$$u = (k\sin\theta)\rho$$

$$\frac{du}{k\sin\theta} = d\rho$$

$$= \frac{1}{(k\sin\theta)^2} \cdot u J_1(u)\Big|_0^{Rk\sin\theta} = \frac{R}{k\sin\theta} \cdot J_1(kR\sin\theta)$$

Squaring this:

$$|f(\theta)|^2 = k^2 \left(\frac{R}{k\sin\theta}\right)^2 \left[J_1(kR\sin\theta)\right]^2$$

$$= \frac{R^2}{\sin^2\theta}\left[J_1(kR\sin\theta)\right]^2$$

$$\frac{d\sigma}{d\Omega} = (\pi R^2)\, \frac{1}{\pi} \left[\frac{J_1(kR\sin\theta)}{\sin\theta}\right]^2$$

Classical
X-section

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$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta$$

$$J_n'(x) = \frac{1}{\pi}\int_0^\pi \left[-\sin(n\theta - x\sin\theta)\right](-\sin\theta)\, d\theta$$

Trig identity:

$$2\sin(n\theta - x\sin\theta)\sin\theta = \cos\left[(n\theta - x\sin\theta) - \theta\right]$$

$$- \cos\left[(n\theta - x\sin\theta) + \theta\right]$$

$$= \cos\left[(n-1)\theta - x\sin\theta\right] - \cos\left[(n+1)\theta - x\sin\theta\right]$$

so

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$