Solutions to PS 9

1. Recall d'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left[u_0(x-t) + u_0(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)\,ds$$

where

$$u(x, t=0) = u_0(x)$$

$$\partial_t u(x, t=0) = u_1(x)$$

9.6.1 $u_0(x) = \sin x$

$u_1(x) = \cos x$

$$u(x,t) = \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \cos(s)\,ds$$

$$= \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right] + \frac{1}{2}\left[\sin(x+t) - \sin(x-t)\right]$$

$$= \sin(x+t)$$

this is a left moving wave: $x = -t$

Check $u(x, t=0) = \sin x$

$\partial_t u(x, t=0) = \cos x$

When $c \neq 1$ see next page

9.6.2 $u_0(x) = \delta(x)$

$u_1(x) = 0$

$$u(x,t) = \frac{1}{2}\left[\delta(x-t) + \delta(x+t)\right]$$

superposition of a left and right moving wave.

PS 9-1*

**Note** Suppose $c \neq 1$ in 9.6.1;

When $c \neq 1$,

$$u(x,t) = \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct)\right]$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(s)\,ds$$

$$u(x,t) = \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\left[\sin(x+ct) - \sin(x-ct)\right]$$

$$= \left(\frac{1+c}{2c}\right)\sin(x+ct) + \left(\frac{c-1}{2c}\right)\sin(x-ct)$$

$$= \sin x \cos ct + \frac{1}{c}\sin ct \cos x$$

9.6.4. $u_0(x) = 0$

$u_1(x) = \sin x$

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \sin s \, ds = -\frac{1}{2}\left[\cos(x+t) - \cos(x-t)\right]$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

so $\cos(x+t) - \cos(x-t) = -2\sin x \sin t$

$\Rightarrow \quad u(x,t) = \sin x \sin t$ .

Check $\quad u(x,t=0) = 0 = u_0(x)$

$\partial_t u(x,t=0) = \sin x = u_1(x)$

and

$\partial_t^2 u(x,t) = -u(x,t)$

$\partial_x^2 u(x,t) = -u(x,t)$

so $\left(\partial_t^2 - \partial_x^2\right) u(x,t) = 0$ .

PS9-3

2. 14.7.5

Show: $j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x} j_n(x)$

Use $j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$ and

the result of 14.1.8:

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

Take $\nu = n + \frac{1}{2}$ so

$$J_{n-\frac{1}{2}}(x) + J_{n+\frac{3}{2}}(x) = \frac{2(n+\frac{1}{2})}{x} J_{n+\frac{1}{2}}(x)$$

$$\Rightarrow \sqrt{\frac{\pi}{2x}} J_{n-\frac{1}{2}}(x) + \sqrt{\frac{\pi}{2x}} J_{n+\frac{3}{2}}(x) = \frac{2n+1}{x} \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$$

$$\Rightarrow \boxed{j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x} j_n(x)}$$

Show:

$$n j_{n-1}(x) - (n+1) j_{n+1}(x) = (2n+1) j_n'(x)$$

First, use the $2^{nd}$ result of 14.1.8:

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_\nu'(x)$$

with $\nu = n + \frac{1}{2}$

$$(*) \quad J_{n-\frac{1}{2}}(x) - J_{n+\frac{3}{2}}(x) = 2J'_{n+\frac{1}{2}}(x)$$

Second, compute $j_n'(x) = \frac{d}{dx}\left(\sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)\right)$

(B9-4)

$$j_n'(x) = \frac{1}{2}\left(\frac{\pi}{2x}\right)^{-\frac{1}{2}}\left(\frac{-\pi}{2x^2}\right) J_{n+\frac{1}{2}}(x) + \sqrt{\frac{\pi}{2x}}\, J'_{n+\frac{1}{2}}(x)$$

$$= \frac{1}{2}\left(\frac{2x}{\pi}\right)\left(\frac{-\pi}{2x^2}\right) j_n(x) + \sqrt{\frac{\pi}{2x}}\, J'_{n+1/2}(x)$$

$$= -\frac{1}{2x} j_n(x) + \sqrt{\frac{\pi}{2x}}\, J'_{n+\frac{1}{2}}(x)$$

Third, multiply $(*)$ by $\sqrt{\frac{\pi}{2x}}$ :

$$j_{n-1}(x) - j_{n+1}(x) = 2\sqrt{\frac{\pi}{2x}}\, J'_{n+\frac{1}{2}}(x)$$

$$= 2\left[ j_n'(x) + \frac{1}{2x} j_n(x)\right]$$

so $j_{n-1}(x) - j_{n+1}(x) - \frac{1}{x} j_n(x) = 2 j_n'(x)$

This isn't quite the desired result but use the previous identity to get:

$$j_{n-1}(x) - j_{n+1}(x) - \frac{1}{2n+1}\left(j_{n-1}(x) + j_{n+1}(x)\right) = 2 j_n'(x)$$

$$= 2n\, j_{n-1}(x) - (2n+2)\, j_{n+1}(x) = 2(2n+1)\, j_n'(x)$$

so

$$\boxed{n\, j_{n-1}(x) - (n+1)\, j_{n+1}(x) = (2n+1)\, j_n'(x)}$$

P9.5

14.7.6. Show $j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$

By induction:

$\underline{n=0}$ $\quad j_0(x) = \left(\frac{\sin x}{x}\right)$

Assume true for $n$ & write for $(n+1)$

Recursion formula $\quad -x^{-n} j_{n+1}(x) = \frac{d}{dx}\left[x^{-n} j_n(x)\right]$

$$j_{n+1}(x) = -x^n \frac{d}{dx}\left[\frac{1}{x^n}(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)\right]$$

$$= (-1)^{n+1} x^{n+1} \left(\frac{1}{x}\frac{d}{dx}\right)^{n+1} \left(\frac{\sin x}{x}\right)$$

as required.