MA/PHY 507 Spring 2019
Midterm Exam - 100 Points
8 March 2019

INSTRUCTIONS: PLEASE WORK ALL FOUR PROBLEMS BELOW. NO BOOKS, PAPERS, OR NOTES ARE ALLOWED.

NAME: Solutions

PROBLEM	MAXIMUM GRADE	SCORE
1	25	
2	25	
3	25	
4	25	
TOTAL	100	

Problem 1. (25 points.) Laurent expansions, poles, and residues:

i. Find the Laurent expansion of f about $z_0 = 0$. Identify the order of the pole, the residue, and the region where the expansion is valid.

$$f(z) = \frac{2z^2}{(z+1)^2}$$

ii. Identify the poles and their orders and calculate the residues for the function

$$f(z) = \frac{e^z}{e^{2z}-1}.$$

(i) Singularity at $z_0 = -1$ so expect LE in $(z - z_0) = (z+1)$

$f(z) = \frac{1}{(z+1)^2} \cdot 2z^2$ only need to express $2z^2$ in terms of $(z+1)$: $z^2 = (z+1-1)^2 = (z+1)^2 - 2(z+1) + 1$

so $f(z) = \dfrac{2(z+1)^2 - 4(z+1) + 2}{(z+1)^2} = 2 - 4\dfrac{1}{z+1} + \dfrac{2}{(z+1)^2}$

$= \dfrac{2}{(z+1)^2} - 4\dfrac{1}{z+1} + 2$

$\text{Res}(f,-1) = -4$; $z_0 = -1$ is a 2^{nd} order pole.

You can also use $\lim_{z \to -1}\left(\frac{d}{dz}(z+1)^2 f(z)\right) = \text{Res}(f,-1)$

(ii) Poles at $e^{2z} = 1 \Leftrightarrow e^{2x}\cos 2y = 1$ (1)

$e^{2x}\sin 2y = 0 \Rightarrow \sin 2y = 0$ or $2y = \pi k$, $k \in \mathbb{Z}$

so $y = \frac{\pi k}{2}$. Substitute into (1) $e^{2x}\cos \pi k = (-1)^k e^{2x} = 1 \Rightarrow x = 0$, $k = 2m$ even

so $(x = 0, y = m\pi)$, Note that e^z doesn't vanish at these points.

let $h(z) = e^{2z} - 1$, $h'(z) = 2e^{2z}$, and $g(z) = e^z = g'(z) \neq 0$.

$\text{Res}(f, im\pi) = \dfrac{g(im\pi)}{h'(im\pi)} = \dfrac{(-1)^m}{2} = \dfrac{(-1)^m}{2}$.

All the poles are simple.

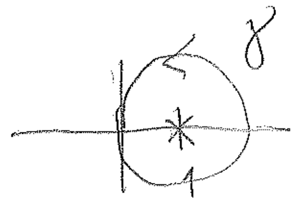
Problem 2. (25 points.) Compute the following integrals. Clearly explain your methods.

i.

$$\oint_{|z-1|=1} \frac{e^{z^2}}{(z-1)^3}\, dz$$

ii.

$$\oint_{|z|=3} \frac{z}{z^2+2z+5}\, dz$$



(i) $f(z) = e^{z^2}$ entire; γ contains $z_0 = 1$

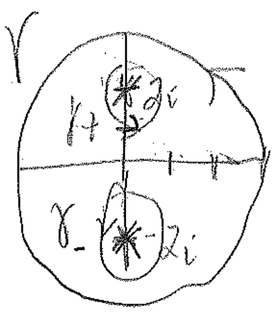
Cauchy's formula with $n=2$

$$f^{(2)}(1) = \frac{2}{2\pi i} \oint \frac{f(z)}{(z-1)^3}\, dz = \left[\frac{d^2}{dz^2} e^{z^2}\right]_{z=1}$$

$$= \left[e + 2z^2 e^{z^2}\right]_{z=1}$$

$$= 6e.$$

$$\Rightarrow \boxed{\oint_{|z-1|=1} \frac{e^{z^2}}{(z-1)^3}\, dz = 6e\pi i}$$



(ii) $z^2 + 2z + 5 = 0 \Rightarrow z_\pm = \dfrac{-2 \pm [4-20]^{\frac{1}{2}}}{2} = 1 \pm 2i$

both simple poles inside γ

Deform γ to $\gamma_+ \cup \gamma_-$

$$\oint_{|z|=3} \frac{z\, dz}{(z-z_+)(z-z_-)} = \oint_{\gamma_+} \frac{(z/z-z_-)}{z-z_+} + \oint_{\gamma_-} \frac{\frac{z}{z-z_+}}{z-z_-}\, dz$$

$$= 2\pi i \left(\frac{z_+}{z_+-z_-} + \frac{z_-}{z_--z_+}\right) = 2\pi i$$

You can also use the Residue theorem

$$\oint_\gamma \frac{z}{z^2+2z+5}\, dz = 2\pi i \left[\operatorname{Res}(f, z_+) + \operatorname{Res}(f, z_-)\right]$$

$$= 2\pi i \left[\frac{z_+}{z_+-z_-} + \frac{z_-}{z_--z_+}\right] = 2\pi i$$

Problem 3. (25 points.)

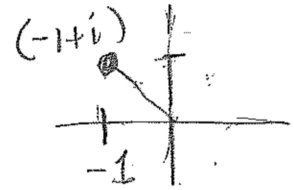
i. Compute the real and imaginary parts of $(-1+i)^{4i}$ using (1) the principal branch, and (2) the branch determined by $[\pi, 3\pi)$.

ii. Is the function $u(x, y) = y^3 - 3x^2y$ the real part of an analytic function? Why or why not? If so, find such an analytic function.

(i.) $(-1+i)^{4i} = e^{4i\log(-1+i)}$. Recall $\log z = \log|z| + i \arg z$

PB $\arg z \in [-\pi, \pi)$.

$\arg_{PB}(-1+i) = \frac{3\pi}{4}$



2nd Branch $\arg z \in [\pi, 3\pi)$

$\arg_2(-1+i) = \frac{11\pi}{4}$

$|z| = \sqrt{2}$.

$\Rightarrow \log_{PB}(-1+i) = \frac{1}{2}\log 2 + \frac{3\pi}{4} i$ and $\log_2(-1+i) = \frac{1}{2}\log 2 + \frac{11\pi}{4} i$

$\Rightarrow (-1+i)^{4i} = e^{-3\pi}\left(\cos(2\log 2) + i \sin(2\log 2)\right)$ in PB

$(-1+i)^{4i} = e^{-11\pi}\left(\cos(2\log 2) + i\sin(2\log 2)\right)$ in Branch 2

(ii) $u(x,y) = y^3 - 3x^2y$. $\partial_x u = -6xy$, $\partial_x^2 u = -6y$; $\partial_y u = 3y^2 - 3x^2$, $\partial_y^2 u = 6y$ $\Big\}$ $(\partial_x^2 + \partial_y^2)u = 0$

u harmonic.

CR $\Rightarrow$

$\partial_x u = \partial_y v$ so $v = \int dy(\partial_x u) + g(x)$

$= -3xy^2 + g(x)$ Check: $\partial_y v = -6xy = \partial_x u$ ✓

$\partial_y u = -\partial_x v$ so $v = -\int dx(\partial_y u) + h(y)$ Check: $\partial_x v = -3y^2 + 3x^2$

$= -3y^2x + x^3 + h(y)$ $= -[\partial_y u]$ ✓

Compare: $v(x,y) = -3y^2x + x^3 + C$ $\boxed{f = iz^3 + D}$

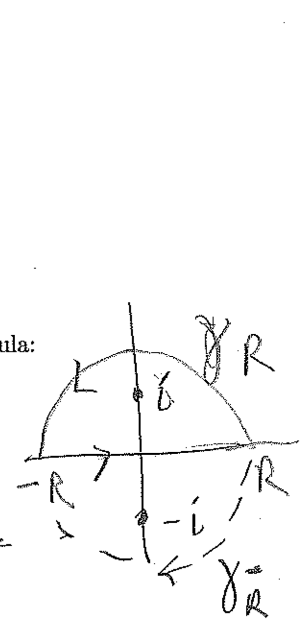
For $f = u + iv = (y^3 - 3x^2y) + i(x^3 - 3xy^2) + D = (x+iy)^3 i + D$

Problem 4. (25 points.) Compute the following real integral:

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx.$$

There are at least two ways to do this. One method might need the formula:

$$\mathrm{Res}(f, z_0) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}.$$



$(x^2+1)^2=0 \iff x^2+1=0 \iff x=\pm i$

Close the contour $\int_{-R}^{R} f(x)\,dx$ in a semi-circle in the <u>Upper</u> half $\mathbb{C}$- plane

$$\oint_{\gamma_R} \frac{z^2}{(z^2+1)^2}\,dz = 2\pi i\ \mathrm{Res}(f,+i) = 2\pi i\left(\frac{-i}{4}\right) = \frac{\pi}{2}$$

If you use the lower half $\mathbb{C}$-plane γ_R^- the orientation is. <u>CW</u> ; $-\int_{\gamma_R^-} f(z)\,dz = 2\pi i\ \mathrm{Res}(f,-i)$.

<u>$\mathrm{Res}(f,i)$</u> : let $g(z)=z^2 \quad g'(z)=2z$

$h(z)=(z^2+1)^2 \quad h'(z)=2(z^2+1)2z$

$h''(z)=4(z^2+1)+8z^2$

$h'''(z)=24z$

$$\mathrm{Res}(f,i) = \frac{2i}{-8} - \frac{2}{3}\frac{(-1)24i}{64} = -\frac{i}{2}+\frac{i}{4} = -\frac{i}{4}$$

[Easier : write $\oint_{\gamma_R} \frac{z^2/(z+i)^2}{(z-i)^2}\,dz = 2\pi i\left[\frac{d}{dz}\frac{z^2}{(z+i)^2}\right]_{z=i} = \frac{\pi}{2}$.]

Finally! $\lim_{R\to\infty}\left|\int_0^{\pi}\left(\frac{R^2e^{2i\theta}}{(R^2e^{2i\theta}+1)^2}\right)Rie^{i\theta}\,d\theta\right| \leq \lim_{R\to\infty}\frac{R^3}{(R^2-1)^2} = 0.$