MA533 Partial Differential Equations
Fall 2011
Problem Set 1
September 2, 2011
DUE: Friday, 9 September

(1) (Evans, pg. 85 #1). Find an explicit formula for the solution $u : \mathbb{R}^n \to \mathbb{R}$ of the initial value problem:

$$\begin{aligned} u_t + b \cdot Du + cu &= 0, \ (x,t) \in \mathbb{R}^n \times (0, \infty) \\ u(x, t=0) &= g(x), \ (x, t=0) \in \mathbb{R}^n \times \{0\}. \end{aligned} \tag{1}$$

Here, the vector $b \in \mathbb{R}^n$ and the number $c \in \mathbb{R}$ are constants.

(2) (Evans, pg. 85, #2). Suppose that $u(x)$ is a solution to Laplace's equation $\Delta u = 0$ on $\mathbb{R}^n$. Prove that if O is an orthogonal transformation, then the function $v(x) \equiv u(Ox)$ is also a solution to Laplace's equation.

(3) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1-boundary. Prove the following identities using the Gauss-Green Theorem:

- Integration by parts: For any $u, v \in C^1(\overline{\Omega})$, we have

$$\int_\Omega u_j v = - \int_\Omega u v_j + \int_{\partial\Omega} u v \nu_j, \tag{2}$$

where $\nu(x)$ is the outward normal vector at $x \in \partial\Omega$.

- Green's formula: For any $u, v \in C^2(\overline{\Omega})$, we have

$$\int_\Omega u \Delta v = - \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} (\nu \cdot \nabla v) u. \tag{3}$$

- For any $u, v \in C^2(\overline{\Omega})$, we have

$$\int_\Omega [u \Delta v - v \Delta u] = \int_{\partial\Omega} [u(\nu \cdot \nabla v) - v(\nu \cdot \nabla u)]. \tag{4}$$