MA533 Partial Differential Equations
Fall 2011
Problem Set 3
September 26, 2011
DUE: Monday, 3 October 2011

(1) Evans, page 86, # 7, an explicit Harnack inequality for the ball.

(2) Prove the converse of the Mean Value Theorem: If $u \in C^2(U)$ satisfies the MVT, then u is harmonic in U, using the following representation. For any $u \in C^2(U)$ and for any $x \in U$, we have

$$(\Delta u)(x) = \lim_{r \to 0} \frac{2n}{r^2} \left[\frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x + r\omega)\, dS(\omega) - u(x) \right].$$

HINT: Use a Taylor expansion with remainder (see problem 5 on page 13) and note that $\int_{\partial B(0,1)} x_i = \int_{\partial B(0,1)} x_j x_k = 0$ for $j \neq k$.

(3) I mentioned that we can modify the fundamental solution $\Phi(x)$ instead of cutting away a small ball about the singularity. For example, for $n \geq 3$, we can define a smooth function on $\mathbb{R}^n$ by:

$$\Phi_\epsilon(x) = \frac{C_n}{[|x|^2 + \epsilon^2]^{(n-2)/2}}, \ \epsilon > 0.$$

The constant $C_n = [\alpha(n)n(n-2)]^{-1}$. Prove that for any $f \in C_0^\infty(\mathbb{R}^d)$, we have

$$\lim_{\epsilon \to 0} \int (-\Delta \Phi_\epsilon)(x) f(x)\, dx = f(0).$$

This is a distributional way of stating that $-\Delta\Phi(x) = \delta(x)$, where $\delta(x)$ is the Dirac delta function (a distribution). This means that for any nice function f, formally we have $\int_{\mathbb{R}^n} \delta(x) f(x)\, dx = f(0)$. Technically, a distribution is a linear functional on a family of nice functions. You can do the same for dimension $n = 2$. HINT: To do the radial integral, use the substitution $s = u^2(1 + u^2)^{-1}$.