MA533 Partial Differential Equations
Fall 2011
Problem Set 7
November 21, 2011
DUE: Wednesday, 30 November 2011

(1) Evans, page 89, # 21, second edition, showing that the components of the electric and magnetic fields satisfy the wave equations, and the relation between the Lamé equation and the wave equation.

(2) The point of this problem is to explore spherically symmetric solutions to the wave equation (WE) in higher dimensions.

(a) Suppose u solves the WE and is spherically symmetric: $u(x,t) = u(r,t)$, with $r = |x| > 0$. Then u solves:

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0. \tag{1}$$

Set $U(r,t) = r^{\alpha} u(r,t)$. What conditions on $\alpha \geq 0$ and the dimension $n \geq 2$ guarantees that U satisfies the WE on $\mathbb{R}_+ \times (0, \infty)$.

(b) Show that the general spherically symmetric solution to the three-dimensional WE has the form

$$u(r,t) = \frac{F(r+t) + G(r-t)}{r},$$

for functions F and G on $\mathbb{R}$.

(c) Suppose that u is a spherically symmetric solution to the WE in three-dimensions with initial data of the form

$$u(r,0) = 0, \; u_t(r,0) = g(r),$$

where g is an even function on $\mathbb{R}$. Show that

$$u(r,t) = \frac{1}{2r} \int_{r-t}^{r+t} s g(s) \; ds.$$

(3) Consider the two-dimensional WE with compactly supported data g and h. Using Poisson's formula for the solution, prove that $u(x,t) \to 0$ as $t \to \infty$ for any fixed $x \in \mathbb{R}^2$. This is in accordance with experience: if you drop a stone into a shallow puddle of water, the spherical waves disperse to infinity.