MA575
Fall 08

Solns. to the Final

F.1

1. $f:[a,b]\to\mathbb{R}$ bdd. $f\in\mathcal{R}([a,b]) \iff \exists A$ s.t. $\forall \varepsilon>0$ $\exists P_\varepsilon$ partition of $[a,b]$ s.t. $|U(f,P_\varepsilon)-A|<\varepsilon$ & $|L(f,P_\varepsilon)-A|<\varepsilon$.

<u>Proof</u> $\Rightarrow$ Take $A=\int_a^b f$. Since for any partition P

$$L(f,P)\le \underline{\int_a^b} f = \overline{\int_a^b} f \le U(f,P) \quad (*)$$

given $\varepsilon>0$ $\exists P_\varepsilon^{(1)}$ s.t. $|A-L(f,P_\varepsilon^{(1)})|<\varepsilon$ (since $\operatorname{lub}_P L(f,P)=A$)

Similarly $\exists P_\varepsilon^{(2)}$ s.t. $|A-U(f,P_\varepsilon^{(2)})|<\varepsilon$. (since $\operatorname{glb}_P U(f,P)=A$)

Take P_ε = refinement of $P_\varepsilon^{(1)}$ & $P_\varepsilon^{(2)}$ then the condition holds.

$\Leftarrow$ For P_ε as stated $|U(f,P_\varepsilon)-L(f,P_\varepsilon)|\le|U(f,P_\varepsilon)-A| + |L(f,P_\varepsilon)-A|<\varepsilon$ so the fundamental criterion is satisfied and $f\in\mathcal{R}([a,b])$. Another way to see this is to note that by $(*)$

$$A-\varepsilon\le L(f,P_\varepsilon)\le\underline{\int_a^b} f\le\overline{\int_a^b} f\le U(f,P_\varepsilon)\le A+\varepsilon$$

for any ε so $\underline{\int_a^b} f=\overline{\int_a^b} f = A$. ∎

F-2

2. Use the criterion: f on $[a,b]$ is in $\mathcal{R}([a,b])$ iff $\forall \varepsilon>0$ $\exists$ partition P_ε of $[a,b]$ s.t. $|U(f,P_\varepsilon) - L(f,P_\varepsilon)| < \varepsilon$.

Pf $f:[a,b]\to\mathbb{R}$ monotone incr. For any $P=\{x_0=a<x_1<\dots<x_n=b\}$

$$U(f,P)-L(f,P)=\sum_{i=0}^{n-1}\underbrace{(f(x_{i+1})-f(x_i))}_{\text{nonnegative by monotonicity}}(x_{i+1}-x_i)\geq 0$$

Given $\varepsilon>0$:

If $f(b)-f(a)=0$ f is a constant so cont. on $[a,b]$ so integrable

If $f(b)-f(a)>0$, let P_ε be the partition of uniformly spaced x_i with $\Delta x = \dfrac{\varepsilon}{f(b)-f(a)}$

$$0\leq U(f,P_\varepsilon)-L(f,P_\varepsilon)\leq\left(\frac{\varepsilon}{f(b)-f(a)}\right)\left[\sum_{i=0}^{n-1}f(x_{i+1})-f(x_i)\right]=\frac{\varepsilon}{f(b)-f(a)}(f(b)-f(a))$$
$$=\varepsilon$$

since the sum is telescoping. Note: If f is strictly monotone incr, then one can choose a partition with $n=[f(b)-f(a)]/\varepsilon$ so $f(x_{i+1})-f(x_i)=\varepsilon/b-a$. We obtain x_i by $f^{-1}(f(x_i))$ (i.e. partition $[f(a),f(b)]$). then $U(f,P_\varepsilon)-L(f,P_\varepsilon)\leq\frac{\varepsilon}{b-a}\cdot b-a=\varepsilon$. This only works when f is strictly monotone.

F-3

4\. 1) Use the Weierstrass M-test: $\left|\frac{\cos^n x}{n^2}\right| \le \frac{1}{n^2} \equiv M_n$

$\sum_{n=1}^{\infty} M_n$ conv. (by p-test) so the original series conv. unif on $\mathbb{R}$

2.) Recall: $|r|<1$, $\sum_{j=0}^{n} r^j = \frac{1-r^{n+1}}{1-r}$. To see this, let $\tilde{S}_n = \sum_{j=0}^{n} r^j$

so $\tilde{S}_{n+1} = \tilde{S}_n + r^{n+1} = r\tilde{S}_n + 1$. Apply with $r = \frac{1}{1+x^2}$ $(x \neq 0)$

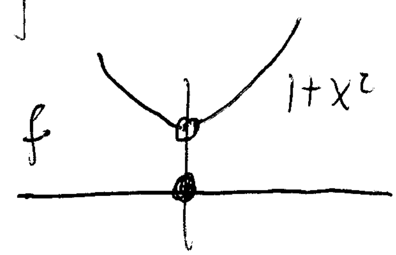
Then:

$$S_n(x) = x^2 \sum_{j=0}^{n} \frac{1}{(1+x^2)^j} = \frac{x^2\left[1 - \frac{1}{(1+x^2)^{n+1}}\right]}{\left[1-\frac{1}{1+x^2}\right]} = (1+x^2) - \frac{1}{(1+x^2)^n}$$

$x \neq 0$

$S_n(x=0) = 0$ by direct calc.

Limits $\lim_{n\to\infty} S_n(x) = \begin{cases} 0 & x = 0 \\ 1+x^2 & x \neq 0 \end{cases}$



f is discont. so the series can't conv. unif. on $\mathbb{R}$

But, on any $I \subset \mathbb{R}\setminus\{0\}$ compact

$$|S_n(x) - S_m(x)| = \left|\frac{1}{(1+x^2)^n} - \frac{1}{(x^2+1)^m}\right|$$

so we have unif conv.

3\. Since f is cont, $f \in R([a,b])$ and on any subinterval $[a,x]$.

Set $F(x) \equiv \int_a^x f$. Compute: $\left|\frac{1}{h}[F(x+h)-F(x)] - f(x)\right| \le \frac{1}{h}\int_x^{x+h} |f(s)-f(x)|$.

Given $\varepsilon > 0$ $\exists\, \delta_\varepsilon > 0$ s.t. $|x-s| < \delta_\varepsilon \Rightarrow |f(s)-f(x)| < \varepsilon$ so

$h < \delta_\varepsilon \Rightarrow \left|\frac{1}{h}\left(F(x+h)-F(x)\right) - f(x)\right| < \frac{1}{h}\cdot h \cdot \varepsilon = \varepsilon$. This

means $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$ exists, so $F'(x)$ exists, and equals

$f(x)$, $x \in (a,b)$.

(F-4)

5. $C_R([0,1])$ separable. We know polynomials are dense in $C_R([0,1])$. Given $\varepsilon>0$, $f \in C_R([0,1])$ $\exists$ $p(x)$, a real polynomial so $\|f-p\|_\infty < \varepsilon/2$. Let k be the order of p so $p(x) = \sum_{n=0}^{k} a_n x^n$, $a_n \in \mathbb{R}$, $x \in [0,1]$. For each a_j, $j=0,1,\dots,k$, $\exists$ $q_j \in \mathbb{Q}$ so $|a_j - q_j| < \varepsilon/2(k+1)$ (density of $\mathbb{Q}$ in $\mathbb{R}$). Then $|p(x) - \sum_{j=0}^{k} q_j x^j| \le \sum_{j=0}^{k} |a_j - q_j| < \varepsilon/2$ so if $q(x) = \sum_{j=0}^{k} q_j x^j$, $\|p-q\|_\infty < \varepsilon/2$.
Hence, $\|f-q\|_\infty < \varepsilon$. This shows the set Q of polynomials with rational coef. is dense in $C_R([0,1])$. This is a countable set as it is the union of all k-tuples of $\mathbb{Q}$s.
For $C([0,1])$ take polynomials with complex coef. $z_j = a_j + i b_j$ and $a_j, b_j \in \mathbb{Q}$.

6. $f_n \xrightarrow{unif} f$ on I & $x_n \to x_0$. Since each f_n is cont. & the conv. is unif, f is cont. Since $I \subset \mathbb{R}$ is compact, each f_n and f is uniformly continuous. We must show: given $\varepsilon > 0$ $\exists$ N_ε s.t. $n > N_\varepsilon \Rightarrow |f_n(x_n) - f(x_0)| \le \varepsilon$.

Pf Given $\varepsilon > 0$:

$$f_n(x_n) - f(x_0) = \underbrace{[f_n(x_n) - f_m(x_n)]}_{1} + \underbrace{[f_m(x_n) - f(x_n)]}_{2} + \underbrace{[f(x_n) - f(x_0)]}_{3}$$

1 $\{f_n\}$ unif Cauchy $\exists$ $N_1(\varepsilon)$ s.t. $n, m > N_1(\varepsilon) \Rightarrow \|f_n - f_m\|_\infty \le \varepsilon/3$ so
$$|f_n(x_n) - f_m(x_n)| \le \|f_n - f_m\|_\infty \le \varepsilon/3.$$
2 $f_n \to f$ uniformly $\exists$ $N_2(\varepsilon)$ s.t. $m > N_2(\varepsilon) \Rightarrow \|f_m - f\|_\infty \le \varepsilon/3$ so
$$|f_m(x_n) - f(x_n)| \le \|f_m - f\|_\infty \le \varepsilon/3.$$
3 f is unif cont. on I so $\exists$ $N_3(\varepsilon)$ s.t. $|x-y| < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon/3$. Now $x_n \to x_0$ so $\exists$ $N_3(\varepsilon)$ s.t. $n > N_3(\varepsilon) \Rightarrow |x_n - x_0| < \delta(\varepsilon)$. Consequently, $n > N_3(\varepsilon) \Rightarrow |f(x_n) - f(x_0)| \le \varepsilon/3$.
Set $N(\varepsilon) \equiv \max(N_1(\varepsilon), N_2(\varepsilon), N_3(\varepsilon))$ so $n > N(\varepsilon)$ then 1 - 3 imply $|f_n(x_n) - f(x_0)| < \varepsilon$. ∎