MA 575 Solutions to PS 1

pg.5 4. $S_1 = 1 - \left(\frac{1}{2}-\frac{1}{3}\right) - \left(\frac{1}{4}-\frac{1}{5}\right) - \left(\frac{1}{6}-\frac{1}{7}\right) - \cdots = 1 - \sum_{n=2}^{\infty} a_n$

$$a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0 \quad \text{positive } n = 2, 4, \ldots$$

In fact, one has $0 < \sum_{n=2}^{\infty} a_n < \int_2^{\infty} \frac{dx}{x(x+1)} = \log\frac{3}{2} < 1$

so $0 < S_1 < 1 - \log 3/2 < 1$.

pg.9 1. If $m-n \equiv m'-n'$ & $p-q \equiv p'-q'$ then we know

$$(*) \begin{cases} m+n' = m'+n \\ p+q' = p'+q \end{cases} \quad m, n, m', n', p, q, p', q' \in \mathbb{N}$$

Now by (14),

$$(m-n)+(p-q) = (p+m)-(n+q)$$
$$(m'-n')+(p'-q') = (m'+p')-(n'+q')$$

so $(m-n)+(p-q) \equiv (m'-n')+(p'-q')$

$\Leftrightarrow (m+p)+(n'+q') = (m'+p')+(n+q)$

This latter statement follows by add the 2 eqns $(*)$.

13. No $r \in \mathbb{Q}$ s.t. $r^2 = 3$. Let $r = p/q$, $p, q \in \mathbb{Z}$ and that 3 doesn't divide both p & q. Then $r^2 = 3 \Leftrightarrow p^2 = 3q^2$ so 3 divides p. Write $p = 3m$ $m \in \mathbb{Z}$ so $9m^2 = 3q^2 \Rightarrow 3m^2 = q^2$ so 3 divides q. Contradiction!

pg.12 1. S & S' 2 cuts. These satisfy (C1) - (C3). Show $\subset$ is an ordering on the set of cuts.

Either $S = S'$ or $S \neq S'$. If $S = S'$, we've done

If not, $S \neq S'$. If $\exists\, s' \in S'$, $s' \notin S$ then

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for $s \in S$, either $s < s'$ or $s' < s$ (ordering of $\mathbb{Q}$)
If $s' < s$ (C2) $\Rightarrow s' \in S$ contradiction
so $s < s' \Rightarrow S \subset S'$. Otherwise, $\exists\, s \in S$
s.t. $s \notin S'$ and the proof $\Rightarrow S' \subset S$. ∎

pg 74. 1. $a, b \in \mathbb{R}$. $a = b \Leftrightarrow \forall \varepsilon > 0$, $|a-b| < \varepsilon$.

Pf $\Rightarrow$ $a = b \Rightarrow |a-b| = 0 < \varepsilon \;\; \forall \varepsilon > 0$.

$\Leftarrow$ If $|a-b| < \varepsilon \;\forall \varepsilon > 0$, suppose $a \neq b$. Then $\exists$ $\delta > 0$ s.t. $a = b + \delta$ or $b = a + \delta$. Either way, $|a-b| = \delta$. If $\varepsilon < \delta$, then $|a-b|$ isn't $< \delta$ a contradiction. Thus $\delta = 0$ & $a = b$. ∎.