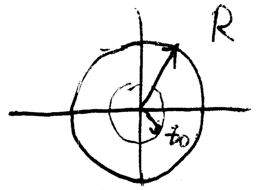
Problem Set #10 Solutions

Pg 66 #2 $f(z) = (1-z)^{-2} = \frac{d}{dz}(1-z)^{-1} = \frac{d}{dz}\sum_{j=0}^{\infty} z^j$ for $|z|<1$

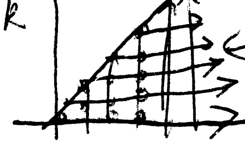
$= \sum_{j=0}^{\infty} j z^{j-1} = \sum_{j=1}^{\infty} j z^{j-1} = \sum_{j=0}^{\infty} (j+1) z^j$. Conv if $|z|<1$, unif on closed subdisks.

#5. $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ If R is the radius of conv. for $\sum_j a_j z^j$ then as long as $|z-z_0|<R$ the series will conv. unif. on subdisks about z_0. To rewrite the series use:



$|z_0| = r < R$

$$\sum a_j (z-z_0)^j = \sum_{j=0}^{\infty} a_j \sum_{k=0}^{j} \binom{j}{k} (-z_0)^{j-k} z^k$$



Look at the sums — first sums over vertical intervals — fix j and sum $k=0\ldots j$. $\sum_{j=0}^{\infty}\sum_{k=0}^{j} B_{jk}$

This sum is the same as $\sum_{k=0}^{\infty}\sum_{j=k}^{\infty} B_{jk}$ i.e. sum over horizontal half-lines

Then
$$\sum_{k=0}^{\infty}\sum_{j=k}^{\infty} \binom{j}{k} (-z_0)^{j-k} z^k a_j = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \binom{m+k}{k} (-z_0)^m a_{m+k} z^k$$
let $m = j-k$

$$= \sum_{k=0}^{\infty} \Big(\underbrace{\sum_{m=0}^{\infty} \binom{m+k}{k} (-z_0)^m a_{m+k}}_{b_m} \Big) z^k$$

Study $b_k = \sum_{m=0}^{\infty} \binom{m+k}{k} \left(\frac{-z_0}{r+\varepsilon}\right)^m \left[(r+\varepsilon)^{m+k} a_{m+k}\right] \frac{1}{(r+\varepsilon)^k}$

Since $r+\varepsilon < R$ $\sum_k (r+\varepsilon)^{m+k} |a_{m+k}| < \infty$ so $(r+\varepsilon)^{m+k}|a_{m+k}|$ is bounded

$$|b_k| < \sum_{m=0}^{\infty} \binom{m+k}{k} \left(\frac{|z_0|}{r+\varepsilon}\right)^m \frac{1}{(r+\varepsilon)^k} = \frac{(r+\varepsilon)^k}{(2r+\varepsilon)^k}\frac{1}{(r+\varepsilon)^k} = \frac{1}{(2r+\varepsilon)^k}$$

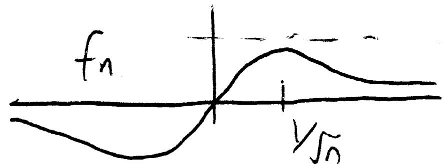
so $|b_k|$ is bounded and the series $\sum b_n$ conv. abs.

2. Let $\{f_n\}$ be unif. conv. with $\|f_n\|_\infty < M_n$ $\forall n$. Then as the seq is unif. Cauchy given $\varepsilon > 0$ $\exists N_\varepsilon$ s.t. $\|f_n - f_m\| < \varepsilon$ $\forall n, m > N_\varepsilon$. Fix $m > N_\varepsilon$ so $\|f_n\| \le \varepsilon + \|f_m\|$ $\forall n > N_\varepsilon$

If we let $N_1 = \max\{\|f_1\|, \dots, \|f_n\|\}$
then $\|f_t\| \le \max(N_1, \varepsilon + \|f_m\|)$ $\forall t$, so
$\|f_n\| \le M$ $\forall n$ & the seq is uniformity bdd.

3. $f_n(x) = \dfrac{x}{1+nx^2}$ $\quad \|f_n\|_\infty \le \dfrac{1}{2\sqrt{n}}$ $\forall n$ so unif. bdd.

$f_n'(x) = (1+nx^2)^{-1} - (1+nx^2)^{-2}\, 2nx^2$



Max at $\frac{1}{\sqrt{n}}$ & $f_n(\frac{1}{\sqrt{n}}) = \frac{1}{2\sqrt{n}}$

$f_n(x) \to 0$ ptw and as $\|f_n - 0\|_\infty \le \dfrac{1}{2\sqrt{n}}$, $f_n \to 0$ unif.

Now $f_n'(x) = \dfrac{1 - nx^2}{(1+nx^2)^2}$ $\quad f_n'(0) = 1$ so $f_n'(0) \to 1$ } but $x \neq 0$ $\lim f_n'(x) = 0$

so $f_n'(x) \to \begin{cases} 1 & x = 0 \\ 0 & \text{other} \end{cases}$ pointwise and not to $f'(x) = 0$.

4. Write $f_n g_n - fg = (f_n - f) g_n + f(g_n - g)$.
Since each $\{f_n\}$ & $\{g_n\}$ are uniformly bdd by #2
($\|f_n\| \le F$ $\forall n$, $\|g_n\| \le G$ $\forall n$) we know
$\big| \|f_n\| - \|f\| \big| \le \|f_n - f\| \to 0$ so $\|f\| \le F$, $\|g\| \le G$.
Given $\varepsilon > 0$ let $N_\varepsilon^{(1)}$ be s.t. $n > N_\varepsilon^{(1)} \Rightarrow$
$\|f_n - f\| < \dfrac{\varepsilon}{2G}$ and $N_\varepsilon^{(2)}$ s.t. $m > N_\varepsilon^{(2)} \Rightarrow$
$\|g_n - g\| < \dfrac{\varepsilon}{2F}$. From * for $n \ge \max(N_\varepsilon^{(1)}, N_\varepsilon^{(2)})$

$\|f_n g_n - fg\| \le \|f_n - f\| G + \|g_n - g\| F \le \varepsilon$.