MA575 Solutions to PS 2 (2-1)

pg 20 #4. (a) $A, B \subset \mathbb{R}$ bdd above. Then (i) $\inf(-A) = -\sup A$ and (ii) $\sup(A+B) = \sup A + \sup B$.

4 pts

Pf (i) A bdd above so $\exists\, k \geq x\ \forall x \in A \Rightarrow -x \geq -k$ so $-A$ is bdd below. Let $a = \sup A$ so $-x \geq -a\ \forall x \in A$. Suppose $-a < b$ we show b can't be a lower bound on $-A \Rightarrow -a = \inf(-A) = -\sup A$. Suppose $-x \geq b\ \forall x \in A$ then $x \leq -b < a$ so $-b$ is an UB on A less than a, a contradiction since $a = \sup A$.

(ii) Since $x \leq \sup A,\ \forall x \in A$, $y \leq \sup B\ \forall y \in B \Rightarrow$ $x+y \leq \sup A + \sup B$ so $\sup A + \sup B \in UB(A+B) \Rightarrow$ $\sup(A+B) \leq \sup A + \sup B$. For the other inequality, note that for $y \in B$ fixed, $x + y \leq \sup(A+B)$ so $\sup(A+B)$ is an $UB(A + \{y\}) \Rightarrow \sup A + y \leq \sup(A+B)$. Now vary $y \in B$ so $\sup A + \sup B \leq \sup(A+B)$. ∎

NB If $A + c = \{x + c \mid x \in A\}$ $\sup(A+c) = \sup A + c$.

pg 20 #5. $I_n = [a_n, b_n]$, $a_n \leq b_n$, $I_{n+1} \subset I_n\ \forall n$, and $|I_n| = b_n - a_n \to 0$. Then $\exists!\, x$ s.t. $x \in \bigcap_n I_n$.

3 pts.

Pf Uniqueness: suppose $c, d \in \bigcap_n I_n \Rightarrow c, d \in I_n\ \forall n$ $\Rightarrow |c-d| \leq |I_n| < \varepsilon\ \forall n > N_\varepsilon \Rightarrow c = d$.

Existence: $a_1 \leq a_2 \leq \dots \leq b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1$ (*)

$\{a_j\}$ monotone incr bdd above $a = \text{lub } a_j$

$\{b_j\}$ monotone decr. bdd below $b = \text{glb } b_j$

both exist.

because of (*) $b, a \in \bigcap_n I_n \Rightarrow a = b$ by uniqueness.

pg 20 #4 Part (b). GLB Property: $A \subset \mathbb{R}$ bdd below has a glb.

PS 2 (2-2)

Pf $A \subset \mathbb{R}$ nonempty & bdd below $\Rightarrow$ $-A$ nonempty & bdd above so $\text{lub}(-A) = a$ exists. Claim $-a = \text{glb}\,A$.
$\forall x \in A$, $-x \le a$ so $x \ge -a \Rightarrow -a \in LB(A)$.
If $y > -a$ is a $LB(A) \Rightarrow -y < a$ is an $UB(-A)$
(note: $y \in LB(A) \Rightarrow x \ge y\ \forall x \in A$ so $-x \le -y$)
This contradicts the fact that $a = \text{lub}(-A)$.
Hence, $-a = \text{glb}(A)$. ∎

pg 28 #5. $z \neq 0$ complex. $\exists\,(r, w)$ unique $r > 0$, $|w| = 1$ so $z = rw$.
Set $w = \frac{z}{|z|}$ (possible as $|z| \neq 0$), then $|w| = \frac{|z|}{|z|} = 1$.

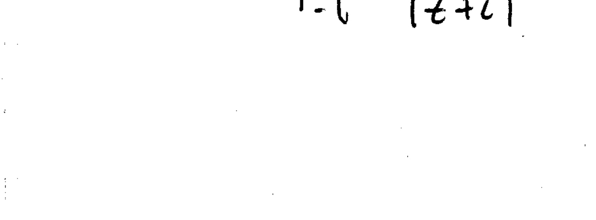
2 pts

Set $r = \bar{w} z = \frac{|z|^2}{|z|} = |z| > 0$, then $wr = w\bar{w}z = |w|z = z$.

If $\exists$ another pair $(\hat{r}, \hat{w})$ w/ $\hat{r}\hat{w} = rw$ taking the modulus of both sides $\Rightarrow \hat{r} = r$ so $r\hat{w} = rw$. As $r \neq 0$, $\hat{w} = w$. ∎

#7. $|z - i| = |z + i| \overset{(*)}{\Leftrightarrow} z \in \mathbb{R}$. The condition means
$x^2 + (y-1)^2 = x^2 + (y+1)^2$ so $(y-1)^2 = (y+1)^2$ which is only possible if $y = 0$. So z satisfying $(*)$ must be real.

1 pt.



distance from z to $\pm i$ equal only when $z \in \mathbb{R}$ (the triangle is an isoscelese triangle)