MA575: Solutions to PS 3

pg 32 1. Suppose $\{x_n\}$ is a bounded real seq. with $a \le x_n \le b$, and $\lim_{n\to\infty} x_n = x$. Then: $a \le x \le b$.

Proof For any $\varepsilon > 0$ $\exists$ N_ε s.t. $n > N_\varepsilon \Rightarrow |x - x_n| < \varepsilon$ so $x - \varepsilon < x_n < x + \varepsilon$. Since $x_n \ge a$ we get $a \le x_n < x + \varepsilon$, and as $x_n \le b$, $x \le b + \varepsilon \Rightarrow a - \varepsilon < x < b + \varepsilon$ for any ε so $a \le x \le b$. ∎

7. If $z_n \to z_0$ and $w_n \to w_0$ then $z_n w_n \to zw$.

Pf $z_n w_n = (z_n - z_0) w_n + z_0 (w_n - w_0) + z_0 w_0$.
Given $\varepsilon > 0$ $\exists$ N s.t. $|z_n - z_0| < \varepsilon/2W$ and $|w_n - w_0| < \varepsilon/2Z$, where W is a bound on $|w_n| \le W$ and Z is a bound on $|z_n| \le Z$.
Then $\forall n > N$:

$$|z_n w_n - z_0 w_0| \le |z_n - z_0| W + |w_n - w_0| Z \le \varepsilon. \;∎$$

10. Show that LUB property follows from the fact that bounded monotone seq. converg.

Proof $A \subset \mathbb{R}$ bounded above. Let $UB(A) = \{$all upper bounds of $A\}$.
This set is bdd below by any element of A.
Choose $r_1 \in UB(A)$ so that $r_1 - 1 \notin UB(A)$.
Let $n_1 \in \mathbb{N}$ smallest s.t. $r_1 - \frac{1}{n_1} \in UB(A)$.

- If no smallest n_1 exists, $r_1 - \frac{1}{n} \notin UB(A)$ $\forall n \in \mathbb{N}$.
  In this case, $r_1 = \text{lub} A$ for if $s < r_1$ is an UB; $r_1 - \frac{1}{n} < s \le r_1$, for all $n \Rightarrow s = r_1$ (if $s < r_1 - \frac{1}{n}$ some $n \Rightarrow r_1 - \frac{1}{n} \in UB(A)$)
- If not, set $r_2 = r_1 - \frac{1}{n_1}$. Let $n_2 \in \mathbb{N}$ smallest s.t. $r_2 - \frac{1}{n_2} \in UB(A)$

Again, if no such n_2 exists, $r_2 = \text{lub} A$ as above.
Continue to construct a monotone, decr. seq.
$r_1 > r_2 > r_3 = r_2 - \frac{1}{n_2}$, $r_k = r_{k-1} - \frac{1}{n_{k-1}}$ bounded
below by any element of A

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Set $r_\infty = \lim_{k\to\infty} r_k$. Claim: • $r_\infty \in UB(A)$. If $\exists\, a \in A$ s.t. $a > r_\infty$, then as $\varepsilon = a - r_\infty > 0$ $\exists\, j$ s.t. $r_\infty < r_j < a$ contradicting the fact that $r_j \in UB(A)$.

• $r_\infty = \text{lub } A$. If $\exists\, s < r_\infty$ an $UB(A)$, then let $\varepsilon = r_\infty - s > 0$ $\exists\, j$ s.t. $|r_\infty - r_j| < \varepsilon/2$ and $r_{j-1} - \frac{1}{n_{j-1}} \in UB(A)$ but

$r_{j-1} - \frac{1}{n_{j-1}-1} \notin UB(A)$. Now $\left| r_{j-1} - \frac{1}{n_{j-1}-1} - \left(r_{j-1} - \frac{1}{n_{j-1}} \right) \right| = \left| \frac{1}{n_{j-1}} - \frac{1}{n_{j-1}-1} \right|$

$= 1 / n_{j-1}(n_{j-1}-1) \to 0$ $j \to \infty$ so $r_{j-1} - \frac{1}{n_{j-1}-1} > s$

Since $r_{j-1} - \frac{1}{n_{j-1}-1} \notin UB(A)$ this contradicts the claim that

$s < r_\infty$ is an UB. ∎