pg 40 1. <u>Prop. 3.12</u> Let $\{x_n\}$ be a real, bdd seq.

(a) If $\{x_{n_i}\}$ is a conv. subseq then: $\liminf x_n \le \lim x_{n_i} \le \limsup x_n$

<u>pf</u> $\limsup x_{n_i} = \lim_{i\to\infty} \sup\{x_{n_i}, x_{n_{i+1}}, \dots\}$

Note that $\{x_{n_i}, x_{n_{i+1}}, \dots\} \subset \{x_{n_i}, x_{n_i+1}, x_{n_i+2}, \dots\}$

so $\sup\{x_{n_i}\, x_{n_{i+1}} \dots\} \le \underbrace{\sup\{x_{n_i}, x_{n_i+1}, \dots\}}$

$\Rightarrow \lim x_{n_i} = \limsup x_{n_i} \le \limsup x_n$ subseq of the seq. that converges to $\limsup x_n$.

Similarly for the lim inf.

$\inf\{x_{n_i}, x_{n_{i+1}}, \dots\} \ge \inf\{x_{n_i}, x_{n_i+1}, \dots\}$

so

$\lim x_{n_i} = \liminf x_{n_i} \ge \liminf x_n$. ∎

You can also use Prop. 3.3.

pg 48. <u>Th. 4.5</u> $b_n \ge 0$ and $\exists$ cnst. $M, N > 0$ s.t. $|a_n| \le M b_n$ $\forall n \ge N$. If $\sum b_n < \infty$ then $\sum a_n$ conv. abs.

<u>pf</u> Let $s_n = \sum_{j=1}^n |a_j|$ & $t_n = \sum_{j=1}^n b_j$ be the partial sums.

Since t_n is a monotone incr seq bdd above $t_n \to T < \infty$. (by hypothesis). We also have

$$0 \le s_n \le \underbrace{\left(\sum_{j=1}^N |a_j|\right)}_{A_0 \text{ finite}} + \sum_{N+1}^n M b_n \quad (n > N)$$

$$\le A_0 + M\left(\sum_{N+1}^n b_j\right) + 2M\left(\sum_{j=1}^N b_j\right)$$

$$\le B_0 + M t_n \le B_0 + MT$$

So s_n is a monotone incr. seq. bdd above and hence converges. ∎

MA 575 Problem Set 4

pg 38 2. **Prop 3.10** $\{b_n\}$ real bdd seq. and $a_n > 0$ with $\lim_n a_n = a$. Then $\liminf a_n b_n = a \liminf b_n$.

Proof let's make 2 observations:

(1) $\liminf (x_n + y_n) \geq \liminf (x_n + y_n)$ (see pg 35 #3,4)

(2) $\liminf (\lambda b_n) = \lambda \liminf b_n$, $\lambda > 0$.

Given these, then

$$a_n b_n = (a_n - a) b_n + a b_n$$

so (1) $\Rightarrow$

$$\liminf a_n b_n \geq \liminf (a_n - a) b_n + a \liminf b_n$$

now $(a_n - a) \to 0$ and $|b_n| < B < \infty$ (bounded)

$$\lim_{n\to\infty} |(a_n - a) b_n| = 0 = \liminf (a_n - a) b_n$$

giving $\liminf a_n b_n \geq a \liminf b_n$.

To prove the other inequality, write

$$a b_n = (a_n - a_n) b_n + a_n b_n$$

so $\liminf a_n b_n \geq \liminf a_n b_n$.

Proof of (1):

$$\liminf (x_n + y_n) = \lim_{n\to\infty} \left[\inf \{ x_n + y_n, x_{n+1} + y_{n+1}, \dots \} \right]$$

let X_n be the set of all $\{x_j\}_{j=n}^{\infty}$ & Y_n the set of all $\{y_\ell\}_{\ell=n}^{\infty}$

We know: $\inf (X_n + Y_n) = \inf X_n + \inf Y_n$. (pg 20 #4)

and if $A \subset B$ then $\inf A \geq \inf B$

Since $\{ x_n + y_n, \dots \} \subset X_n + Y_n = \{ x_j + y_m \mid j, m = n, \dots \infty \}$

$$\inf \{ x_n + y_n, \dots \} \geq \inf (X_n + Y_n) = \inf X_n + \inf Y_n$$

so $\liminf (x_n + y_n) \geq \liminf x_n + \liminf y_n$

Proof of (2): A bounded below then $\inf \lambda A = \lambda \inf A$, $\lambda > 0$.

and $\liminf \lambda x_n = \lim \lambda \inf (x_n, x_{n+1}, \dots) = \lambda \liminf x_n$