MA575 Problem Set 5

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2. $\sum_{n=1}^{\infty}\left(\frac{3^n}{n^3}\right)z^n \qquad a_n = \frac{3^n}{n^3} \qquad \frac{a_n}{a_{n+1}} = \frac{3^n}{n^3}\cdot\frac{(n+1)^3}{3^{n+1}} = \left(\frac{n+1}{n}\right)^3\frac{1}{3}$

$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3\frac{1}{3} = \frac{1}{3} \qquad R = 1/3$

6. $\sum_{n=1}^{\infty}\frac{n!}{n^n}z^n \qquad a_n = \frac{n!}{n^n} \qquad \frac{a_n}{a_{n+1}} = \frac{n!}{(n+1)!}\;\frac{(n+1)^{n+1}}{n^n}$

$= \frac{1}{n+1}\left(\frac{n+1}{n}\right)^n(n+1) = \left(1+\frac{1}{n}\right)^n = e^{n\log(1+\frac{1}{n})}$

Compute: $\lim_{x\to\infty} e^{x\log(1+\frac{1}{x})}$. Use l'Hopital's rule: $\lim_{x\to\infty} x\log(1+\frac{1}{x}) = \lim_{x\to\infty}\frac{\log(1+\frac{1}{x})}{\frac{1}{x}}$. This is a $\frac{0}{0}$ indeterminate form so

$$\lim_{x\to\infty} x\log(1+\tfrac{1}{x}) = \lim_{u\to 0}\frac{\log(1+u)}{u}$$
$$= \lim_{u\to 0}\frac{\frac{1}{1+u}}{1} = 1$$

So by continuity $\lim_{n\to\infty} e^{n\log(1+\frac{1}{n})} = e$.

$R = e$.

10. $\sum_{n=0}^{\infty} a_n z^n$ has radius R and $0<r<R$,

(a) $\exists K$ s.t. $|z|\le r$, $\left|\sum_{n=0}^{\infty} a_n z^n\right| \le K$.

<u>Pf</u> $S_n(z) = \sum_{j=0}^{n} a_j z^j$ so $|S_n(z)| \le \sum_{j=1}^{n} |a_j||z|^j \equiv t_n(z)$.

If $|z|<R$, the PS converges absolutely so $\lim_{j\to\infty} t_n(z) =$ $t(z)$ exists & is finite. If $|z| = r < R$, the seq $t_n(z) \to t(z)$ ...

and $|s_n(z)| \leq t_n(|z|=r) \leq t_\infty(|z|=r)$ by monotonicity

Hence, $\left|\sum_{j=0}^{\infty} a_j z^j\right| \leq t_\infty(|z|=r)$ since we know

$\lim_{n\to\infty} s_n(z) = s_\infty(z)$ exists for $|z|<R$.

(b) For any $k \in \mathbb{N}$ $\exists$ K_k s.t. if $|z| \leq r < R$, $\left|\sum_{j=k}^{\infty} a_j z^j\right| < K_k |z|^k$

Pf For any $n > k$,
$$\left|\sum_{j=k}^{n} a_j z^j\right| \leq |z|^k \sum_{j=k}^{n} |a_j| \, |z|^{j-k}$$

since $|z| \leq r < R$ and $j-k \geq 0$, $|z|^{j-k} \leq r^{j-k}$

so
$$\left|\sum_{j=k}^{n} a_j z^j\right| \leq |z|^k \cdot \frac{1}{r^k}\left(\sum_{j=k}^{n} |a_j| r^j\right)$$

By part (a) $\lim_{n\to\infty} \sum_{j=k}^{n} |a_j| r^j = t_\infty$ exists $(t_\infty \leq t_\infty(|z|=r))$

and $\lim_{n\to\infty} \sum_{j=k}^{n} a_j z^j = s_{\infty,k}(z)$ exists

so
$$\left|\lim_{n\to\infty} \sum_{j=k}^{n} a_j z^j\right| = \left|\sum_{j=k}^{\infty} a_j z^j\right| \leq |z|^k K_k$$

with $K_k \equiv \frac{1}{r^k}\left(\sum_{j=k}^{\infty} |a_j| r^j\right)$ ∎

pg. 74 1. $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

- pos. defn. $d_1(x,y) \geq 0$, $= 0$ iff $x_i = y_i$ $\forall i$
- symmetric $d_1(x,y) = d_1(y,x)$
- triangle ineq:

$$|x_i - y_i| \leq |x_i - w_i| + |w_i - y_i|$$

so $d_1(x,y) \leq d_1(x,w) + d_1(w,y)$.

$d_\infty(x,y) = \sup_{j=1,\dots,n} |x_i - y_i|$ (sup is a max)

- pos. defn. $d_\infty(x,y) \geq 0$

if $d_\infty(x,y) = 0 \Rightarrow$ max at j

$0 \leq |x_i - y_i| \leq |x_j - y_j| = 0$

$\Rightarrow x_i = y_i$

- symm.
- $\sup |x_i - y_i| \leq \sup(|x_i - w_i| + |w_i - y_i|) \leq \sup |x_i - w_i| + \sup |w_i - y_i|$

Equivalence:

$n\, d_1(x,y) \geq n \cdot d_\infty(x,y) \geq d_1(x,y)$

so $\frac{1}{n} d_1(x,y) \leq d_\infty(x,y) \leq d_1(x,y)$.

3. $(X, \|\cdot\|)$ normed LVS. Define $d(x,y) = \|x - y\|$ $\forall x, y \in X$

d is well defined as X is a LVS.

- d is pos. defn. Obviously $d(x,y) \geq 0$. If $x = y$, $d(x,x) = 0$. If $d(x,y) = 0 = \|x - y\|$ then the pos defn property of $\|\cdot\|$ implies $x = y$.
- d is symm $\|x - y\| = \|y - x\|$ by homog. of the norm.
- triangle ineq: $d(x,y) = \|x - y + w - w\|$

$$\leq \|x - w\| + \|w - y\|$$
$$\leq d(x,w) + d(w,y). \quad \forall x, w, y \in X.$$