MA575

(6-1)

Solutions to PS 6.

pg 81 #1a. (S, d_S) metric sp. & $A \subset S$ finite. Then A is compact. Let $\mathcal{U} = \{\mathcal{O}_\alpha\}$ be an open cover of A: $\bigcup_\alpha \mathcal{O}_\alpha \supset A$, $\mathcal{O}_\alpha$ open. Since $A \subset S$ is finite, $A = \{a_1, \ldots, a_K\}$ some $K \in \mathbb{N}$ finite & $a_j \in S$. Each $a_j \in \mathcal{O}_{\alpha_j}$ so $\{\mathcal{O}_{\alpha_j}\}_{j=1}^{K}$ is a finite subcover of $\mathcal{U}$: $A \subset \bigcup_{j=1}^{K} \mathcal{O}_{\alpha_j}$, so A is compact. ∎

b. Suppose d_S is the discrete metric: $d_S(x,x) = 0$ and $d_S(x,y) = 1$ $\forall y \neq x$. If $A \subset S$ is compact then A is finite. Note that $N_\varepsilon(x)$ - ε-ball in the d_S metric topology - has the property that $N_\varepsilon(x) = \{x\}$ if $\varepsilon \leq 1$ and $N_\varepsilon(x) = S$ if $\varepsilon > 1$, so singetons $\{x\}$ are open sets. Construct an open cover of A: $\mathcal{U} = \{N_\varepsilon(x) \mid x \in A, \varepsilon < 1\}$. Since A is compact, $\exists$ finite subcover $\{N_\varepsilon(x_j) \mid x_j \in A, j = 1, \ldots, M\}$. Since $N_\varepsilon(x_j) = \{x_j\}$ and $A \subset \bigcup_{j=1}^{M} \{x_j\} = \{x_1, \ldots, x_M\} \Rightarrow A$ is finite ∎

c. (S, d_S) be a nonfinite set with the discrete metric. S is closed & bdd but not compact (a compact set must be finite). S is closed becouse it contains all its limit pts. & S is bdd because if $x \in S$, $\{y \mid d_S(x,y) < 2\} = S$. ∎

2. Prop: The union of a finite # of compact sets in a metric sp (X, d_X) is compact. Pf Let $\{A_j\}_{j=1}^{N}$ be N compact sets. Let $A = \bigcup_{j=1}^{N} A_j \subset X$ and let $\mathcal{U}$ be an open cover of A. Then $\mathcal{U}$ covers each A_j so $\exists$ finite subcovers $\mathcal{U}_j = \{\mathcal{O}_{m,j}\}_{m=1}^{n_j}$ with $A_j \subset \bigcup_{m=1}^{n_j} \mathcal{O}_{m,j}$. Then $\bigcup_{j=1}^{N} \mathcal{U}_j$ is a finite open cover of A. Hence any open cover of A has a finite subcover ∎

pg 89. #1. If f is cont. at p & $\lim p_n \to p$ then $\lim f(p_n) = f(p)$. Pf f cont. at p means for any $\varepsilon > 0$ $\exists$ $\delta_p(\varepsilon) > 0$ s.t. $|p - s| < \delta_p(\varepsilon) \Rightarrow |f(p) - f(s)| < \varepsilon$. Since $\lim p_n = p$, given $\delta_p(\varepsilon)$ $\exists$ $N_{\delta_p(\varepsilon)} > 0$ s.t. $n > N_{\delta_p(\varepsilon)} \Rightarrow |p_n - p| < \delta_p(\varepsilon)$ so $|f(p_n) - f(p)| < \varepsilon$ This means $f(p_n)$ conv. to $f(p)$. ∎

3. $f: S \to T$ & $g: T \to U$. Suppose f cont. at $p \in S$ & g cont. at $f(p) \in T$ Then $g \circ f$ is cont. at $p \in S$. Pf Given $\varepsilon > 0$ since g is cont.

at $f(p)$ $\exists\ \delta_\varepsilon(f(p))>0$ s.t. $|t-f(p)|<\delta_\varepsilon(f(p))$ $(t\in T) \Rightarrow |g(f(p))-g(t)|<\varepsilon$.
Now as f is cont. at p, given $\delta_\varepsilon(f(p))$ $\exists\ \sigma_\varepsilon(p)$ s.t. $|s-p|<\sigma_\varepsilon(p)$ $\Rightarrow |f(s)-f(p)|<\delta_\varepsilon(f(p))$. So, given $\varepsilon>0$, if $|s-p|<\sigma_\varepsilon(p)$ $(s\in S)$ then $|g(f(p))-g(f(s))|<\varepsilon$ (taking $t=f(s)$). ∎

7. $B\subset S$ not closed, then $\exists\ f: B\to\mathbb{R}$ cont. not obtaining a max and also not bdd. $S=\mathbb{R}$, with Euclidean metric, $B=(0,1]$ so B isn't closed. $f(x)=1/x : B\to[1,\infty)$. f is unbdd (given $M>\mathbb{R}^+$ $\exists\ x_M\in(0,1]$ s.t. $0<x<x_M$, $f(x)>M$ ($x_M=1/M$) and certainly f can't have a max. on B.
8. $B\subset S$ not closed, then $\exists\ f: B\to\mathbb{R}$ cont. but not unif cont. Same ex. as in 7: f isn't unif. cont. Given $\varepsilon>0$ suppose $\exists\ \delta_\varepsilon>0$ s.t. $|x-y|<\delta_\varepsilon \Rightarrow |f(x)-f(y)|<\delta$, $x,y\in(0,1]$. Fix x and take $y=x-\delta/2$ so $|x-y|=\delta/2<\delta$. But then $|f(x)-f(y)|>\dfrac{\delta}{2x^2}>0$ $(y<x \Rightarrow \frac{1}{x}<\frac{1}{y})$. By taking $x=\sqrt{\frac{\delta}{4\varepsilon}} \Rightarrow |f(x)-f(y)|>2\varepsilon$, a contradiction. Hence, f isn't unif. cont.