7.1

MA575: Solutions to PS #7

1a) Application of l'Hopital's Rule: $f(x) = x\log x$ $\quad f'(x) = \log x + 1$

$g(x) = e^x - e \quad g'(x) = e^x$

g is everywhere diff & f is diff on $(0,\infty)$.

$\lim_{x\to 1^\pm} f(x) = 0 = \lim_{x\to 1^\pm} g(x)$ and g' is never zero.

Then $\lim_{x\to 1^\pm} \frac{f(x)}{g(x)} = \lim_{x\to 1^\pm} \frac{f'(x)}{g'(x)} = \lim_{x\to 1^\pm} \frac{\log x}{e^x} + \lim_{x\to 1^\pm} \frac{1}{e^x} = e^{-1}$.

2. $f:\mathbb{R}\to\mathbb{R}$ f diff & $|f'(x)| < M$ $\forall x\in\mathbb{R}$. Then f is unif. cont. on $\mathbb{R}$

<u>Proof</u> For $x,y\in\mathbb{R}$ the Mean Value Th. states $\exists$ c between x & y so $|f(x)-f(y)| < |f'(c)||x-y| < M|x-y|$. Given $\varepsilon>0$ take $\delta = \varepsilon/M$. Then $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$ showing that f is uniformly continuous. ∎

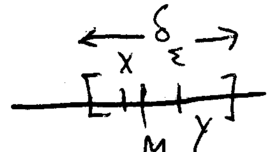
3.a) $f:\mathbb{R}\to\mathbb{R}$ continuous & $\lim_{|x|\to\infty} f(x) = 0$. Then f is unif. cont.

<u>Proof</u> Given $\varepsilon>0$. There is M_ε s.t. $|x|\geq M_\varepsilon \Rightarrow |f(x)| < \varepsilon/2^2$.

For any x,y with $|x|>M_\varepsilon$, $|y|>M_\varepsilon$,

$$|f(x)-f(y)| < |f(x)| + |f(y)| < \varepsilon$$

On $[-M_\varepsilon, M_\varepsilon]\subset\mathbb{R}$, a compact interval, f is uniformly continuous: $\exists\, \delta_\varepsilon>0$ s.t. $x,y\in[-M_\varepsilon, M_\varepsilon]$ w/ $|x-y|<\delta_\varepsilon$ then $|f(x)-f(y)|<\varepsilon/2$. If $x\in[-M_\varepsilon, M_\varepsilon]$ and $y\in\mathbb{R}\setminus[-M_\varepsilon, M_\varepsilon]$ with $|x-y|<\delta_\varepsilon$, then if x is near $+M_\varepsilon$ or $-M_\varepsilon$:



$$|f(x)-f(y)| \leq |f(x)-f(\pm M_\varepsilon)| + |f(\pm M_\varepsilon)| + |f(y)|$$
$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

b) $f:\mathbb{R}\to\mathbb{R}$ bdd, differentiable, unif cont. but f' unbdd.

$f(x) = \frac{\sin(x^4)}{1+x^2}$ then f is cont. $\lim_{|x|\to\infty} f(x) = 0$ so f is unif. cont.

$f'(x) = 4x^3\cos(x^4)(1+x^2)^{-1} - 2x\sin(x^4)(1+x^2)^{-2}$ is unbounded.

Given $M > 0$ let N be an integer $N^{\frac{1}{2}} > M$ then if $x = (2\pi N)^{\frac{1}{4}}$

$f'(x) \geq (2\pi N)^{3/4} > M$ so it is unbounded.

5. f diff (a,b) & $f' \neq 0$ on (a,b). Then let $f(a) = \lim_{x \to a^+} f(x)$ & $f(b) = \lim_{x \to b^-} f(x)$ so f is cont on $[a,b]$. It could happen that these limits are $+\infty$ or $-\infty$, for ex, $f(x) = 1/x$ on $(0,1)$ & $f'(x) \neq 0$ on $(0,1)$ Then work on $(\varepsilon, 1]$ and take $\varepsilon \to 0$. If f has an extrema at $c \in (a,b)$ then $f'(c) = 0$ (simple application of the difference quotient) Since $f'(x) \neq 0$, $x \in (a,b)$ f has <u>no</u> <u>extrema</u> so it is incr or decr. If $f(a)$ is finite, then it may be an extrema: $f(a) < f(b) \Rightarrow$ str. incr etc.

Another approach: if f isn't str. incr. then $\exists\, c$ s.t.

$f(x) < f(c)$ $x < c$ and $f(c) > f(y)$, $y > c$

then $\dfrac{f(c) - f(x)}{c - x} \geq 0$ & $\dfrac{f(c) - f(y)}{c - y} \leq 0$

so $f'(c) = 0$, contradiction.