MA575 Solutions to PS 8

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1. a) \( f: \mathbb{R} \to \mathbb{R} \) and \( |f(x)| \leq x^2 \). Then \( f \) is diff. at \( x = 0 \).

   Pf. For \( x \neq 0 \),
   \[
   \left| \frac{f(x) - f(0)}{x} \right| = \left| \frac{f(x)}{x} \right| \leq |x| \text{ as the bound,}
   \]
   implies \( f(0) = 0 \). Thus \( \lim_{x \to 0} \frac{f(x) - f(0)}{x} = 0 \) and \( f'(0) = 0 \).

   b) Construct \( f: \mathbb{R} \to \mathbb{R} \) the 0 is differentiable at \( x = 0 \). By (a) it suffice to take \( |f(x)| \leq x^2 \) \( \forall x \). Define
   \[
   f(x) = \begin{cases} 
   x^2 & x \in \mathbb{Q} \\
   0 & x \in \mathbb{R}\backslash\mathbb{Q}
   \end{cases}
   \]
   Then \( f \) is diff. at \( x = 0 \). Suppose \( f \) is cont. at \( x_0 \neq 0 \) (it is cont. at \( x_0 = 0 \)) then \( \forall \varepsilon > 0 \) \( \exists \delta > 0 \) s.t. \( |y - x_0| < \delta \Rightarrow |f(y) - f(x_0)| < \varepsilon \).

   Take \( \varepsilon < x_0^2 \).

   Then \( f(y) - f(x_0) < \varepsilon \).

   Choose any \( y \in B(x_0, \delta) \setminus \{x_0\} \), \( y \notin \mathbb{Q} \) (always possible by density of \( \mathbb{Q} \) in \( \mathbb{R} \)) then \( f(y) = 0 \) and \( x_0^2 \leq \varepsilon \), but \( \varepsilon < x_0^2 \) so we get a contradiction so \( f \) is not cont. at any point \( x_0 \neq 0 \).

6. Suppose \( f: [a, b] \to \mathbb{R} \) diff. on \((a, b) \) and \( c \) is such that \( f'(a) < c < f'(b) \) (\( f'(a) \) e \( f'(b) \) are defined as the right & left limits of the difference quotient, resp.). Then \( \exists x \in (a, b) \) s.t. \( f'(x) = c \).

   Pf. Set \( g(x) = (x - f(x)) \). Then \( g'(a) > 0 \), so \( g \) is incr. for \( x \) near \( a \) & \( g'(b) < 0 \) so \( g \) is decr. for \( x \leq b \). In particular, \( g \) isn't constant. Since \( g \) is cont. on \([a, b] \) \( \exists x_0 \in (a, b) \) where \( g(x_0) \) is a max \( \Rightarrow g'(x_0) = 0 \), so \( c = f'(x_0) \). The max can't take place at \( x = a \) since \( f(x) > f(a) \) for \( x > a \) nor can it take place at \( x = b \) since \( f(x) > f(b) \) for \( x < b \).)

pg 112 1a) \( f: [a, b] \to \mathbb{R} \) cont. \( \exists \int (f(x) \leq \int (b) \). \)

   Pf. Let \( m = \min f(x) \) & \( M = \max f(x) \) so \( m \leq \int f(x) \leq M \). If \( m = f(x_1) \) & \( M = f(x_2) \) for \( x_1, x_2 \in [a, b] \) then by
the Intermediate Value Thm 8.9) either \( \exists x \) between \( x_1 \& x_2 \) 
\[ \text{s.t. } f(x_0) = \int_a^{b} f \quad \text{or} \quad f(x_0)(b-a) = \int_a^{b} f = m = M = f \]
is const. In this case \( (b-a)f(x) = \int_a^{b} f \) by construction.

3. \( f \) cont. \( \geq 0 \) on \( [a,b] \) \( (a+b) \) and \( f(b) = 0 \). Then \( f = 0 \) on \( [a,b] \).

**pf** Suppose \( \int_a^{b} f > 0 \) but \( f \) isn't identically zero. Then \( \max f(x) \geq \frac{f(b)}{2} \) and positive. \( \exists \epsilon > 0 \) s.t. \( \forall x \in [x_0 - \epsilon, x_0 + \epsilon] \), \( f(x) > f(x_0)/2 \) (if \( x_0 \) is an endpoint take \( [a, a + \epsilon] \) or \( [b - \epsilon, b] \), resp.) - (claim: \( 0 < f(x_0)\epsilon < \int_a^{b} f \)) giving a contradiction. (Use Prop. 8.19)

\[ \int_a^{b} f = \int_{x_0 - \epsilon/2}^{x_0 + \epsilon/2} f \]

Note: the continuity of \( f \) is essential - see #4.

4. Let \( g : [0,1] \to \mathbb{R} \) be defined by \( g(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n}, n \in \mathbb{N}, \text{ or otherwise} \\ 0 & \text{otherwise} \end{cases} \)

then \( g \in \mathcal{R}([0,1]) \) and \( \int_{0}^{1} g = 0 \) (\( g \) isn't cont. of course).

**pf** Basic fact: \( f : [0,1] \to \mathbb{R} \) \( f(x) = \begin{cases} 1 & x = \frac{1}{n} \\ 0 & \text{other} \end{cases} \)

then \( f \in \mathcal{R}([0,1]) \) \& \( \int_{0}^{1} f = 0 \).

This extends to finitely many points. Given \( \epsilon > 0 \) take \( N \) s.t. \( \frac{1}{N} < \frac{\epsilon}{2} \). Let \( P_{\epsilon} = \left\{ \left[ \frac{1}{N} \right], \cdots, \left[ \frac{N-1}{N} \right] \right\} \). \( U(f, P_{\epsilon}) < \frac{\epsilon}{2} \), which

\( U(f, P_{\epsilon}) = \text{since there are finitely many jump discontinuities} \)

**pf** Define \( P'_{\epsilon} = P_{\epsilon} \cup \left\{ 0, \frac{1}{N} \right\} \) so \( U(f, P'_{\epsilon}) = U(f, P_{\epsilon}) + \frac{1}{N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \) (note \( L(f, P) = 0 \) for all partitions \( P \))

\( \Rightarrow \int_{0}^{1} f \) exists. It also follows that \( \epsilon \left| \int_{0}^{N} f \right| < \epsilon \Leftrightarrow \int_{0}^{N} f = 0 \).