8-1

MA575 Solutions to PS 8

pg 104 4. a) $f: \mathbb{R} \to \mathbb{R}$ and $|f(x)| \leq x^2$. Then f is diff. at $x=0$.

Pf For $x \approx 0$, $\left|\frac{f(x)-f(0)}{x-0}\right| = \left|\frac{f(x)}{x}\right| \leq |x|$ as the bound implies $f(0)=0$. Thus $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ and $f'(0)=0$. ∎

b) Construct $f: \mathbb{R} \to \mathbb{R}$ that is differentiable at $x=0$. By (a) it suffice to take $|f(x)| \leq x^2$ $\forall x$. Define

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Then f is diff. at $x=0$. Suppose f is cont. at $x_0 \neq 0$ (it is cont. at $x_0=0$) Then $\forall \varepsilon > 0$ $\exists \delta_\varepsilon(x_0)$ s.t. $|y-x_0| < \delta_\varepsilon(x_0) \Rightarrow |f(x_0)-f(y)| < \varepsilon$. Take $\varepsilon << x_0^2$. Choose any $y \in N_{x_0}(\delta_\varepsilon(x_0))$, $y \notin \mathbb{Q}$ (always possible by density of $\mathbb{Q}$ in $\mathbb{R}$) then $f(y)=0$ and $x_0^2 < \varepsilon$ but $\varepsilon << x_0^2$ so we get a contradiction so f is not cont. at any point $x_0 \neq 0$.

6. Suppose $f:[a,b] \to \mathbb{R}$ diff on (a,b) & c is such that $f'(a) < c < f'(b)$ ($f'(a)$ & $f'(b)$ are defined as the right & left limits of the difference quotient, resp.). Then $\exists x \in (a,b)$ s.t. $f'(x) = c$. Pf Set $g(x) = cx - f(x)$. Then $g'(a) > 0$ so g is incr. for x near a & $g'(b) < 0$ so g is decr. for $x \lesssim b$. In particular, g isn't constant. Since g is cont on $[a,b]$ $\exists x_0 \in (a,b)$ where $g(x_0)$ is a max $\Rightarrow g'(x_0) = 0$ so $c = f'(x_0)$ (The max can't take place at $x=a$ since $f(x) > f(a)$ for $x \gtrsim a$ nor can it take place at $x=b$ since $f(x) > f(b)$ for $x \lesssim b$.) ∎

pg 112 1a) $f:[a,b] \to \mathbb{R}$ cont. $\exists x \in [a,b]$ s.t. $\int_a^b f = f(x)(b-a)$.

Pf. Let $m = \min_{x\in[a,b]} f(x)$ & $M = \max_{x\in[a,b]} f(x)$ so $m \leq \frac{\int_a^b f}{b-a} \leq M$

If $m = f(x_1)$ & $M = f(x_2)$ for $x_1, x_2 \in [a,b]$ then by the Intermediate Value Th 8.9 either $\exists\ x_0$ between x_1 & x_2 s.t. $f(x_0) = \frac{\int_a^b f}{(b-a)}$ or $f(x_0)(b-a) = \int_a^b f$ or $m = M \Rightarrow f$ is const. In this case $(b-a)f(x) = \int_a^b f$ by Construction ∎

3. f cont. ≥ 0 on $[a,b]$ $(a \neq b)$ and $\int_a^b f = 0$. Then $f = 0$ on $[a,b]$.

Pf Suppose $\int_a^b f = 0$ but f isn't identically zero. Then $\max f(x) = f(x_0) \neq 0$ and positive. $\exists\ \delta_0 > 0$ s.t. on $[x_0 - \frac{\delta_0}{2}, x_0 + \delta_0/2]$, $x \in [a,b]$ $f(x) > f(x_0)/2$ (if x_0 is an endpoint take $[a, a+\delta_0]$ or, $[b-\delta_0, b]$, resp.). Claim: $0 < \frac{f(x_0)\delta_0}{2} < \int_a^b f$, giving a contradiction. Use Prop. 8.19

$$\int_a^b f = \int_a^{x_0 - \delta_0/2} f + \int_{x_0-\delta_0/2}^{x_0+\delta_0/2} f + \int_{x_0+\delta_0/2}^{b} f \geq \frac{f(x_0)\delta_0}{2}$$ ∎

Note: the continuity of f is essential - see #4.

4. Let $g : [0,1] \to \mathbb{R}$ be defined by $g(x) = \begin{cases} 1 & x = 1/n,\ n = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$

Then $g \in \mathcal{R}([0,1])$ and $\int_0^1 g = 0$ (g isn't cont. of course).

Pf Basic fact: $f : [0,1] \to \mathbb{R}$ $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & \text{other} \end{cases}$ then $f \in \mathcal{R}([0,1])$ & $\int_0^1 f = 0$.

This extends to finitely-many points. Given $\varepsilon > 0$ take N s.t. $\frac{1}{N} < \varepsilon/2$. Let $P_\varepsilon = [\frac{1}{N}, 1], \ldots$ $U(f, P_\varepsilon') < \varepsilon/2$, which $U(f, P_\varepsilon) =$ since there are finitely many jump discontinuities. Refine $P_\varepsilon' = P_\varepsilon \cup \{0\}$ so $U(f, P_\varepsilon) = U(f, P_\varepsilon') + \frac{1}{N} \cdot 1 < \varepsilon/2 + \varepsilon/2 = \varepsilon$ (note $L(f,P) = 0$ for all partitions P) $\Rightarrow \int_0^1 f$ exists. It also follows that $0 \leq \int_0^1 f = \int_0^{1/N} f + \int_{1/N}^1 f < \varepsilon \Rightarrow \int_0^1 f = 0$. ∎