Problem Set #9

pg 94 #2. $A = \{ f \in \mathcal{C}([0,1]) \mid |f(x)| \le 1 \} \subset \mathcal{C}([0,1])$

a) A closed & bounded in $\|\cdot\|_\infty$ - norm. Pf. A bounded means $\exists R > 0$ s.t. $A \subset B_R(0)$ since $0 \in A$. But by defn $f \in A \Rightarrow \|f\|_\infty \le 1$ so $A \subset B_R(0)$ for any $R = 1+\varepsilon$, $\varepsilon > 0$. A is closed. If $f_j \to f$ uniformly, $f \in \mathcal{C}([0,1])$. Given $\varepsilon > 0$ $\exists N_\varepsilon$ s.t. $\|f_j - f\|_\infty < \varepsilon$ $\forall j > N_\varepsilon$. Then $\|f\|_\infty \le \|f - f_j\|_\infty + \|f_j\|_\infty \le 1 + \varepsilon$ for any $\varepsilon > 0$ so $\|f\|_\infty \le 1$.

b.) $f_n(x) = x^n$ No subseq f_{n_j} conv. in $\mathcal{C}([0,1])$. 2 ways to show this: since ptwise $\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$ (*) any subseq must conv ptw. to

the same fnc. Now if $f_{n_j} \to g$ unif. it does so pointwise so on the one hand, unif conv. means g is cont. but by (*) it isn't.
Second way: look at $f_{nm}(x) = x^n - x^m$. This is max at $x_0 = \left(\frac{m}{n}\right)^{\frac{1}{n-m}}$ assuming m is fixed & $n > m$. Then $\sup_{x \in [0,1]} |f_{nm}(x)| = f_{nm}(x_0) \cong \left(\frac{1}{n}\right)^{\frac{1}{n}} \to 1$. This means A is not sequentially compact so by theorem 6.11 it isn't compact. So given any subseq $\|f_{n_j} - f_{n_{j'}}\|_\infty > \frac{1}{2}$ for $n_j, n_{j'}$ large enough.

#3. $U_n = \{ f \in \mathcal{C}([0,1]) \mid |f(0) - f(\tfrac{1}{n})| < 1 \}$, $n \in \mathbb{N}$.

a) $\{U_n\}$ covers A, in fact, it covers $\mathcal{C}([0,1])$. Since $f \in \mathcal{C}([0,1])$ is right cont. at 0, given $\varepsilon > 0$ $\exists \delta_\varepsilon > 0$ s.t. $|f(0) - f(x)| < \varepsilon$ if $|x| < \delta_\varepsilon$. Take n s.t. $0 < \frac{1}{n} < \delta_\varepsilon$ so $|f(0) - f(\frac{1}{n})| < \varepsilon < 1$ and $f \in U_n$.

b) U_n is open. Let $f \in U_n$. For any $g \in \mathcal{C}([0,1])$ write

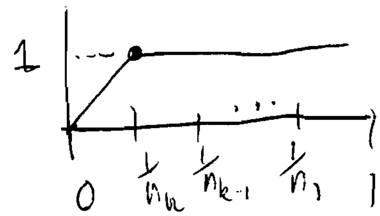
$$|g(0) - g(\tfrac{1}{n})| \le |g(0) - f(0)| + |g(\tfrac{1}{n}) - f(\tfrac{1}{n})| + |f(0) - f(\tfrac{1}{n})| \quad (*)$$

Since $f \in U_n$ $\exists \delta > 0$ s.t. $|f(0) - f(\frac{1}{n})| = 1 - \delta < 1$. For any $\varepsilon > 0$ with $\varepsilon < \delta/4$, and any $g \in N_\varepsilon(f)$, (*) implies $|g(0) - g(\frac{1}{n})| \le 2\varepsilon + 1 - \delta \le 1 - \delta/2 < 1$ so $g \in U_n \Rightarrow N_\varepsilon(f) \subset U_n$ and U_n is open.

c) $\exists$ no finite subcover of $\{U_n\}$ covering A. Suppose $\{U_{n_1}, \dots, U_{n_k}\}$ covers A with $n_1 < n_2 < \dots < n_k$. Let $h(x)$ be:

$$h(x) = \begin{cases} n_k x & 0 \le x \le \frac{1}{n_k} \\ 1 & 0 \le x \le 1 \end{cases}$$

Then $|h(0) - h(\frac{1}{n_k})| = 1$ and $|h(x)| \le 1$ so $h \in A$ but it isn't in any U_{n_j} since $|h(0) - h(\frac{1}{n_j})| = 1$, $j = 1, \dots, k$.



pg. 116. #1. $f \in C_{\mathbb{R}}([0,1])$ $f(0) = f(1) = 0$ $f''(x) \ge 0$ $x \in (0,1)$
Then either $f =$ cnst. or $f(x) < 0$ for $x \in (0,1)$.

pf For $x, y \in (0,1]$, $x > y$, Mean Value Th $\Rightarrow$ $\frac{f'(x) - f'(y)}{x - y} = f''(c_{xy}) \ge 0$ for some $c_{xy} \in (y, x)$. This means f' is non decr. Applied to $x > y = 0$ $f'(x) - f'(0) \ge 0$

By Rolle's Theorem $\exists\, c_0 \in (0,1)$ with $f'(c_0) = 0$.

If $f'(0) > 0$, then $f'(x) > 0$ for all $x > 0$ by strict. so $f(1) - f(0) > 0$ contradicting $f(0) = 0 = f(1)$.

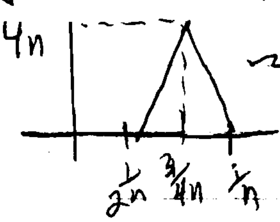
If $f'(0) = 0$ and $f'(x) \ge 0$ (since f' is non decr.) for all x and again we have $f =$ cnst. (or a contradiction)

So $f'(0) < 0$ and as f' is nondecr. $f'(x) < 0$ on $[0, c_0)$.

$\frac{f(x) - \overset{0}{f(0)}}{x} = f'(c_1) < 0$ $c_1 \in (0, c_0)$ so $f(x) < 0$ on $(0, c_0)$

Similarly $f' > 0$ on $(c_0, 1]$ (since f' is nondecr) and $f < 0$.

#8. f_n real & cont. on $[0,1]$



~ tent fnc. with height

Area $= 1 = \int_0^1 f_n$

$f_n(x) \to 0$ ptwise.
f_n is continuous,
Clearly $\lim_{n \to \infty} \int_0^1 f_n = 1$ but $\int_0^1 \lim f_n = 0$ so there is no unif. conv.

#11. $\int_a^b x^n f(x) = 0$ implies for any poly. p: $\int_a^b p(x) f(x) = 0$. Given $\varepsilon > 0$
$\exists\, P_\varepsilon(x)$ poly s.t. $\|f - P_\varepsilon\| < \frac{\varepsilon}{\|f\|(b-a)}$ so $\int_a^b f^2 = \int_a^b f(f - P_\varepsilon) + \underbrace{\int_a^b f P_\varepsilon}_{=0} \le \frac{\varepsilon}{\|f\|(b-a)} \|f\| (b-a)$